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At least, at most revisited*

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Abstract The interaction between modified numerals and modals offers important insights into their interpretation. As pointed out by Büring (2008), not all superlative numerals interacting with modals give rise to the same interpretations: while both *at least* and *at most* interacting with universal modals give rise to *authoritative*/bounded-interpretations, only *at most* interacting with existential modals does so. In the case of interactions with *at least*, the readings are of *speaker-insecurity*/ignorance. This paper offers a new perspective on this interaction, building on a *focus-based* Krifka (1999)-style analysis for superlatives as well as a view of context structured by questions under discussion. The pragmatic component of the proposal makes a case for enrichment driven by considerations of speaker cooperativity and authority. The account thus differs from proposals in the literature that appeal to scalar alternatives and disjunction-style reasoning.

Keywords: Superlative numerals, *at least*, *at most*, modals, authoritative readings, speaker ignorance readings, distributivity inferences, comparative numerals.

1 Introduction

This paper investigates patterns in the availability of readings arising when the superlative numeral modifiers *at least* and *at most* interact with modals, first observed in Büring (2008). Current proposals for the semantics of superlatives often establish links with disjunction, and their interaction with modals can be viewed through this lens. I explore an alternative approach, with a focus-sensitive semantics for superlatives that builds on Krifka (1999) and a link to questions that may be addressed in a discourse.

We find *at least* and *at most* interacting with modals in various kinds of contexts. They can show up in responses to questions that seem to target maximum and minimum amounts, as in (1) and (2):

(1) Scenario: I am planning Alma's tea break, the only time churros are served at my house, and wondering how many to put on her plate (she will eat all of them). Alma's parent has authority over her diet.

   a. Me: How many churros is Alma allowed to have? (What is the maximum amount of churros that Alma is allowed to have?)
   b. Alma's parent: She can have at most two$_F$ churros.

(2) ...

   a. Me: How many churros is Alma is obliged to have? (What is the minimum amount of churros that Alma is obliged to have?)

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b. Alma's parent: Alma must have at least two$_F$ churros.

The *at most*-response in (1b) conveys the upper bound to Alma's permitted churro-eating, while the *at least*-response in (2b) conveys the minimal bound on her obligations. (1b) lets me know that whether Alma has one or two churro(s), the outcome will be within the range of allowed options. More than two churros would cross the line. In the case of (2b), the response lets me know that whether Alma has two/three/ (...) churros, the outcome will be within the range of allowed options. Less than two churros is not allowed. In this scenario, the parent is *authoritative* (they either make the rules or are completely knowledgeable about them). Given the assumption that the parent is authoritative, it would be very strange for them to continue with a sentence that signals epistemic insecurity about what they have just said:

(3) a. Me: How many churros is Alma allowed to have? (What is the maximum amount of churros that Alma is allowed to have?)

b. Alma's parent: She can have at most two$_F$ churros. (# Possibly more / less = It is possible that she can have more/less than two churros.)

The upshot, as observed in Büring (2008), is that *can* + *at most* combinations (1b) and *must* + *at least* combinations (2b) can be understood as setting an upper or lower bound on deontically accessible options (Buring's *authoritative reading* reading). *Authoritative*/ bounded readings have been the focus of much research (a.o. Nouwen (2010), Kennedy (2015), Coppock & Brochhagen (2013), Blok (2019)) and readers are referred to Spector (2020) for an overview.

If, instead of assuming that there is a range of permitted options, we we were to presuppose that there is a unique acceptable option, (1b) [*can*] and (2b) [*must*] would be interpreted as indexing speaker-ignorance (Buring's *speaker-insecurity* reading):

(4) Scenario: For medical reasons, Alma is on an odd diet requiring the consumption of a specific number of churros per day. I don't know what it is, so I ask Alma's parent.

a. Me: How many churros is Alma allowed/obliged to have? (What is the unique amount of churros such that things are well only if Alma has that amount of churros?)

b. Alma's parent: She can have at most two$_F$ churros.

c. Alma's parent: She must have at least two$_F$ churros.

In (4b) and (4c), Alma's parent no longer sounds authoritative. We get the sense that the number of permitted churros was decided by someone else, and the parent had not been paying attention. Ignorance effects with superlatives have also been widely discussed in the literature, both in simple sentences and in interaction with modals (including a.o. Krifka (1999), Geurts & Nouwen (2007), Nakanishi & Rullmann (2009), Nouwen (2010), Coppock & Brochhagen (2013), Penka (2015), Kennedy (2015), Schwarz (2016), Ciardelli, Coppock & Roelofsen (2018), Blok (2019), Mihoc (2019a), Aloni & van Ormondt (2023); see Spector (2020) for overview of the earlier work).

Büring (2008) noted that combinations of *at least* with existential modals fail to have an authoritative reading, obligatorily giving rise to speaker-ignorance inferences:

(5) Buring (2008: 118)

a. The password can be at least 10 characters long. (odd as website instruction)

b. I give you permission to eat at least two candy bars. (plain odd)

In (6b), the *at least* response naturally leads to the inference that the speaker does not know what is the maximum amount of permitted churros.

(6) a. How many churros is Alma allowed to have? (What is the maximum amount of churros that Alma is allowed to have?)
b. She can have at least four churros. (Quite possibly a lot more.)

The absence of an authoritative reading in combinations of *at least* with existential modals has been taken up in later literature (see Spector (2020), also Penka (2015), Buccola & Haida (2018), Blok (2019) a.o.) and I will build on this generalization. It has, however, been questioned, and readers are referred to Kennedy (2015), Coppock & Brochhagen (2013) for alternative views, and to McNabb & Penka (2014) (a.o.) for experimental work.

In what follows I will spell out some background assumptions (Section 2) and then proceed to examine *at least* and *at most* in the context of *every* (Section 3) as a precursor to the discussion of modals (Section 4). I will also establish some connections with comparative numerals *more than* / *less than* and offer concluding remarks (Section 5).

2 Setting the stage

The proposal in this paper makes four assumptions. One is about structure: I will give *at least* and *at most* sentential scope (Section 2.1). One is about semantics (Section 2.2): I will assign *at least* and *at most* focus-sensitive interpretations. The third assumption is about numerals: I will give them an *exactly*-interpretation (Section 2.3). The last one is about context: my account appeals to the *questions in discourse* (Section 2.4).

2.1 Structural assumptions

As a simplifying assumption, I will give both *at least* and *at most* sentential-level (S-level) scope (see Beaver & Clark (2008) for the parallel case of *only*). For a sentence like (7a), the structure to be interpreted is (7b):

(7) a. [Alma had at least two$_F$ churros]
b. [at least [$_S$Alma had two$_F$ churros]]

I will adopt a focus-sensitive interpretation for *at least* and *at most* (see a.o. Krifka (1999)). Like in classic analysis of *only*, *at least* and *at most* end up making claims in relation to focus alternatives. In my proposal, the dependancy between *at least* / *at most* and focus-alternatives will be hard-wired in their interpretation.[1] As we will see in Section 5, *at least* and *at most* differ in this respect from comparative numerals such as *more than* and *less than* whose semantics does not encode dependence on focus alternatives (see Coppock & Brochhagen (2013) for similar contrast).

2.2 Interpreting Superlatives

This section spells out a proposal for *at least* and *at most* that treats them as focus-sensitive particles, building on Krifka (1999). I begin with simple examples without modals, such as (8a) and (8b).

1 Given my current goals, my proposal for superlatives will simply access focus alternatives directly, setting aside possibilities for internal composition in terms of squiggle operators and associated variables. The exact mechanisms are not crucial for my discussion here so I will simplify things in this manner.

(8) a. Alma had at least two$_F$ churros.
b. Alma had at most two$_F$ churros.

As with some other focus-sensitive particles (e.g. *even*), I take utterances with *at least* and *at most* to be understood against the background of ordered focus-alternatives, which I call an *F-scale* (F_{Scale}). Following a.o. Hirschberg (1985), Krifka (1999), Beaver & Clark (2008), Biezma (2011), Coppock & Brochhagen (2013), context and world knowledge are taken to play a role in establishing the relevant scale. I illustrate this in (9), borrowing an example from Beaver & Clark (2008): we are wondering about the celebrity signatures that Alma got at the *Famous Philosophers* party, where Aristotle, Frege, Anscombe and Hegel were in attendance.

(9) a. Alma got at least a Frege$_F$.
b. Alma got at most a Frege$_F$.

As (9) illustrates, the interpretation of superlatives does not depend on an intrinsic ordering amongst the alternatives. While it is possible to contextually recover an ordering between the focus-alternatives *Alma got an Aristotle*, *Alma got a Frege*, etc., there is no relation of strength between the propositions (as Beaver and Clark note, the order could come from the relative prestige of the various philosophers). As long as we are able to reconstruct an ordering for the alternatives, the requirements of the superlatives will be satisfied.

With these clarifications in place, we can now turn to the semantics. We will begin with *at least* where, given an F-scale and world $w \in$ W:

(10) $[\![\textit{at least}\ \text{S}]\!]\ (\text{w}) = 1 \text{ iff } \exists \text{p} \in [\![\text{S}]\!]^{Alt}\text{: p(w)} = 1 \wedge \text{p} \geq_{F_{scale}} [\![\text{S}]\!]$

According to (10), *at least* is sensitive to the set of focus-alternatives of S ($[\![\text{S}]\!]^{Alt}$) and its denotation $[\![\text{S}]\!]$, resulting in a proposition true in a world w iff some alternative in $[\![\text{S}]\!]^{Alt}$ is true in w that is as good as $[\![\text{S}]\!]$ or better given the F-scale associated with the alternatives ($\geq_{F_{scale}}$). I illustrate (10) with (9a): (9a) will receive the interpretation in (11a) with the associated scale in (11b). Given a possible world *w*, the sentence will be true in *w* iff Alma got a Frege in *w* or something higher on the scale:

(11) a. $[\![\text{(9a)}]\!]\text{(w)} = 1 \text{ iff } \exists \text{p} \in [\![\text{Alma got a Frege}_F]\!]^{Alt}\text{: p(w)} = 1$
$\wedge\ \text{p} \geq_{F_{scale}} [\lambda \text{w.Alma got}_w \text{ a Frege}]$
b. $[_{F_{scale}}\ \lambda\text{w.Alma got}_w \text{ an Aristotle} < \lambda\text{w.Alma got}_w \text{ a Frege} < \lambda\text{w.Alma got}_w \text{ a Hegel} <$ $\lambda\text{w.Alma got}_w \text{ an Anscombe ..}\ ]$

At most contrasts with *at least* in terms of its *existential import*. It may be that none of the relevant alternatives is true (see e.g. discussion in Krifka (1999)). An utterance of (9b) is true even if Alma got no autograph. I will capture the lack of existential import with a semantics for *at most* that appeals to universal quantification (a similar proposal can be found in Chen (2024)):[2]

(12) $[\![\textit{at most}\ \text{S}]\!]\ (\text{w}) = 1 \text{ iff } \forall \text{p} \in [\![\text{S}]\!]^{Alt}\text{: p(w)} = 1 \rightarrow \text{p} \leq_{F_{scale}} [\![\text{S}]\!]$.

Given a world w, the interpretation of (9b) would be as in (13), with the same scale as before:

2 Authors who have invoked universal quantification in their characterization of maximality operators associated with *at most* include Coppock & Brochhagen (2013) and Blok (2019).

(13) $[\![(9b)]\!](w) = 1$ iff
$\forall p \in [\![\text{Alma got a Frege}_F]\!]^{Alt}$: $p(w) = 1 \rightarrow p \leq_{F_{scale}} [\lambda w.\text{Alma got}_w \text{ a Frege}]$

(9b) will be true in a world w iff Alma got nothing higher than a Frege in w (i.e. not a Hegel, not an Anscombe).

2.3 Interpreting numerals

The focus-sensitive interpretation for superlatives presented in Section 2.2 is the first of my two semantic moves. The second one is to adopt what is often called an *exactly* semantics for numerals (setting aside matters of internal composition of numeral DPs). Kennedy (2015) argues for such a semantics, that he terms *de-Fregean*, and I will borrow from his implementation (see Spector (2013) and Bylinina & Nouwen (2020) for general discussions of the interpretations of numerals). A simple sentence like (14a) will receive the truth-conditions in (14b), which I will abbreviate as (14c) (given a numeral *N*, I use [*N*] for exactly *N*, [*N*) for *N*/*N*+, and (*N*] for *N*/*N*-, borrowing from Schwarz (2016)):

(14) a. Alma had two churros
b. $[\![(14a)]\!](w) = 1$ iff $max\{n: \exists x\, (\text{had}_w\, (x)(\text{Alma}) \wedge \text{churros}_w\, (x) \wedge \#(x) = n\,)\} = 2$
c. $[\![(14a)]\!](w) = 1$ iff Alma had_w [2] churros

Kennedy (2015) observed that an *exactly*/*de-Fregean* analysis of simple numerals sheds light on ambiguities that arise in their interaction with existential and universal modals (see also Heim (2006), Büring (2008)):

(15) a. Alma can have two churros.
b. Alma must have two churros.

We can interpret (15a) as either making the claim that [2] churros are not ruled out for Alma (i.e. a [2]-churro case would fall within the realm of what is permitted); or the claim that [2] churros is the maximum that is permitted (the upper bound on what is allowed). We can understand (15b) as making the claim that it is required/necessary that Alma have [2] churros (the only permitted option); or the claim that [2] churros is the minimum she must have (a lower bound on what is permitted). Given an *exactly*-semantics for numerals, Kennedy (2015) attributes this ambiguity to scope. In the case of *can* (15a), the modal wide-scope interpretation claims that [2] churros are permitted (16a), while the modal narrow-scope interpretation sets the upper bound on what is permitted to [2] churros (16b):

(16) a. *can* $>$ *two*
$[\![(15a)]\!](w) = 1$ iff
$\exists w' \in \text{Acc}_w$: $max\{n: \exists x\, [\text{had}_{w'}\, (x)(\text{Alma}) \wedge \text{churros}_{w'}\, (x) \wedge \#(x) = n\,]\} = 2$
b. *two* $>$ *can*
$[\![(15a)]\!](w) = 1$ iff
$max\{n: \exists w' \in \text{Acc}_w: \exists x\, [\text{had}_{w'}\, (x)(\text{Alma}) \wedge \text{churros}_{w'}\, (x) \wedge \#(x) = n\,]\} = 2$

In the case of *must* (15b), a modal wide-scope interpretation claims that [2] churros is the only permitted option (17a), while the modal narrow-scope interpretation sets the lower bound of what is permitted to [2] churros (17b):

(17) a. *must* > *two*
$[\![(15b)]\!](w) = 1$ iff
$\forall w' \in Acc_w$: $max\{n: \exists x\ [had_{w'}(x)(Alma) \wedge churros_{w'}(x) \wedge \#(x) = n\ ]\} = 2$

b. *two* > *must*
$[\![(15b)]\!](w) = 1$ iff
$max\{n: \forall w' \in Acc_w: \exists x\ [had_{w'}(x)(Alma) \wedge churros_{w'}(x) \wedge \#(x) = n\ ]\} = 2$

Examples illustrating the interaction of 'exactly' numerals with *at least* and *at most* in simple (non-modal) cases are provided in (18) and (19), with (20) spelling out the relevant scale:

(18) *At least*
$[\![$Alma had at least two$_F$ churros$]\!](w) = 1$ iff
$\exists p \in [\![$Alma had two$_F$ churros$]\!]^{Alt}$: $p(w) = 1 \wedge p \geq_{F_{scale}} [\lambda w.$Alma had$_w$ [2] churros]

(19) *At most*
$[\![$Alma had at most two$_F$ churros$]\!](w) = 1$ iff
$\forall p \in [\![$Alma had two$_F$ churros$]\!]^{Alt}$: $p(w) = 1 \rightarrow p \leq_{F_{scale}} [\lambda w.$Alma had$_w$ [2] churros]

(20) $[_{F_{scale}}\ \lambda w.$ Alma had$_w$ [1] churro $< \lambda w.$ Alma had$_w$ [2] churros $< \lambda w.$ Alma had$_w$ [3] churros $< \lambda w.$ Alma had$_w$ [4] churros ...]

Given the *exactly* interpretation for numerals, the scale in (20) is not ordered in terms of the relative strength of the propositions (there is no entailment relation between the members of the scale). It is, however, possible to recover an order between the propositions in terms of the amounts of churros that Alma had. Just as we did in the case of (9), where we recovered an order between focus alternatives in terms of the relative amount of prestige associated with the philosophers, we can recover an order in (18) and (19) in terms of the relative amount of churros consumed.

2.4 Modelling context

Following a.o. Roberts (2012), Büring (1997), and Beaver & Clark (2008), I will assume that part of the information tracked in a context can be modeled in terms of the objectives of the participants in resolving *questions under discussion*. Somewhat informally, I will characterize contexts in terms of questions as a partition of the set of relevant worlds into the possible answers to the relevant question. Against this background, the idealized role of an assertion is to fully resolve the question in the context. This means that given a set of possible answers to the question, the context-update effect of a fully felicitous/successful assertion will be to discard all but one (only one cell in the partition will 'remain standing' post update). From now on, I will use the term *resolve* in the sense of *fully resolve*. I provide a toy illustration of a context in (21), where (21a) is the question under discussion and (21b) represents the cells in the corresponding partition (each cell corresponds to a possible answer, e.g. the set of relevant possible worlds where Alma had [1] churro, the set where she had [2] churros, etc.):

(21) a. How many churros did Alma have?

b.

Alma had [1] churro	Alma had [2] churros	Alma had [3] churros
Alma had [4] churros	Alma had [5] churros	Alma had [6] churros

An utterance of the sentence *Alma had two$_F$ churros* will update the context, and only one cell will 'survive' in the updated context. In this way, the utterance will resolve the question:

(22)

~~Alma had [1] churro~~	Alma had [2] churros	~~Alma had [3] churros~~
~~Alma had [4] churros~~	~~Alma had [5] churros~~	~~Alma had [6] churros~~

In a context in which we are trying to figure out how many churros Alma had, an utterance of *Alma had at least two$_F$ churros* would fail to resolve the question. The utterance does provide pertinent information (it eliminates one cell in the context), but it leaves open more than one possible answer:

(23)

~~Alma had [1] churro~~	Alma had [2] churros	Alma had [3] churros
Alma had [4] churros	Alma had [5] churros	Alma had [6] churros

Addressing a question while failing to fully resolve it typically leads the listener to infer that the speaker is ignorant. A cooperative speaker who was also knowledgeable would have fully resolved the question. Of course, the ignorance inference is not obligatory, since the listener may be aware of other reasons for the speaker's failure to resolve the question.

(24) a. A: How many churros did Alma have?

b. B: She had at least two. But I am not going to tell you exactly how many. Figure it out yourself.

Given the discussion above, *at least* and *at most* seem at times curiously doomed to failure. In a sentence like *Alma had at least two$_F$ churros*, *at least* will associate with the focus alternatives of its prejacent *Alma had two$_F$ churros*. Such placement of focus will be licensed if there is a question salient in discourse about how many churros Alma had (e.g., Rooth (1992)), a question that the *at least*-sentence will not be able to resolve. Linking a focus-based analysis of *at least* and *at most* with a model of discourse as questions under discussion thus provides insights into the speaker-ignorance inferences often observed with superlative modifiers.

3 Obviating ignorance: the case of *every*

It has been noted in the literature that the ignorance inferences associated with *at least* and *at most* can be obviated in interaction with universal quantifiers (Klinedinst (2007), Büring (2008), Nouwen (2010), Schwarz (2016), Blok (2019), a.o.). This section investigate ignorance obviation in relation to *every* as a preliminary to the discussion regarding modals (Section 4). I will begin by spelling out a discourse-driven account that I base on *authority-driven strengthening* (Section 3.1). I will then briefly sketch an alternative view based on a disjunction-style analysis of superlatives and quantity-based scalar reasoning (Section 3.2), as this will be helpful in the discussion of interactions with modals.

3.1 Authority-driven strengthening

Consider the example in (25a) with the structure in (25b), where the quantificational DP *every linguist* has raised above sentential *at least* to take widest scope:

(25) a. Every linguist had at least two$_F$ churros.

b. [every linguist λ_i [at least [t_i had two$_F$ churros]]].

The structure in (25b) 'distributes' the alternatives amongst the domain of *every*:

(26) $[\![$[every linguist λ_i [at least [t_i had two$_F$ churros]]]$]\!]^g$(w) = 1 iff $\forall$x [linguist$_w$(x) = 1 $\rightarrow$ [$\exists$p $\in$ {λw. x had$_w$ [2] churros, λw. x had$_w$ [3] churros, λw. x had$_w$ [4] churros, ... }: p(w)=1]]

Let us explore the questions that an utterance of (25a) could address/ resolve in a context. It is clear that, given a group of linguists and the assumption that they had different amounts of churros, a universal generalization would not resolve a question about the amount of churros that each of them had. Consider the toy example in (27), a context with three linguists (l_1, l_2 and l_3) and a question regarding the number of churros each had ranging from [3] to [5]:

(27)

l_1: [2] l_2: [5] l_3: [3]	l_1: [5] l_2: [2] l_3: [3]	l_1: [3] l_2: [2] l_3: [5]

The interpretation of (25b) does not discriminate between the cells in (27) and so the corresponding utterance would neither address nor resolve the question (and the addressee might conclude that the speaker was uncooperative).

An utterance of (25b) would, however, address the question in a context in which we had granted that the linguists had eaten a range of number of churros and were wondering about the lowest amount that any of them had. What would such a context look like? In each cell of the partition, we would find worlds where there is a linguist x that had [N] churros, for some number N, and all y different from x that are linguists had [N) churros.

(28)

{w : $\exists$x: x is a linguist$_w$ $\wedge$ x has$_w$ [1] churro $\wedge$ $\forall$y: y is a linguist$_w$ $\rightarrow$ y has$_w$ [1) churro }	{w : $\exists$x: x is a linguist$_w$ $\wedge$ x has$_w$ [2] churros $\wedge$ $\forall$y: y is a linguist$_w$ $\rightarrow$ y has$_w$ [2) churros }	{w : $\exists$x: x is a linguist$_w$ $\wedge$ x has$_w$ [3] churros $\wedge$ $\forall$y: y is a linguist$_w$ $\rightarrow$ y has$_w$ [3) churros }	.. ..

Consider now an utterance of (25b) in this context. The sentence is true if the linguist who had the fewest churros had two, and so the utterance would address the question by eliminating the 'first' cell in (28) (the one corresponding to [1]). By itself, however, this does not resolve the question. The denotation of (25b) leaves open the possibility that the minimum number was [2], that it was [3], or more. If the addressee considers it likely that the speaker of (25b) is not well-informed, things could well stop there, and the utterance would lead to a speaker-ignorance inference by the addressee (*the speaker was cooperative in addressing the question but they didn't resolve it because they couldn't*).

But what if the addressee is convinced that the speaker is knowledgeable/ authoritative and cooperative? Then the addressee may make a further inference. In a context like (28), accepting that the speaker has resolved the question with an utterance of (25b) requires accepting that at least one of the linguists had [2] churros. It is only with this additional assumption that the utterance of (25b) narrows down the possible answers in (28) to one, selecting the second cell (corresponding to [2]) as the answer to the question. So, while there is nothing in the denotation of (25b) that guarantees that some linguist had [2] churros, trust that the speaker is resolving the question can only lead the addressee to infer that at least one linguist did. We can think of this as a kind of *authority-driven* *strengthening* triggered by the premise that the speaker is cooperative: the addressee is willing to take on the additional assumption that some linguist had [2] churros (and thus *enrich* the interpretation of (25b)) to preserve their belief that the speaker is both authoritative and cooperative (*given that the speaker is resolving the question, they must intend to convey that*). Note that crucially, the additional assumption needs to be about the lowest bound: making the assumption that some linguist had N-churros where N > 2 would not have the effect of allowing the utterance to resolve the question (e.g. if the listener made the assumption that some linguist had [3] churros, both the second and third cells in (28) would survive the update). For completeness, I spell out the corresponding example with *at most*:

(29) a. Every linguist had at most two$_F$ churros.
b. [every linguist λ_i [at most [t_i had two$_F$ churros]]]

The sentence in (29b) receives the interpretation in (30):

(30) $[\![(29b)]\!](w) = 1$ iff $\forall x$ [linguist$_w$(x) = 1 $\rightarrow$
[$\forall p \in \{\lambda w.$ x had$_w$ [1] churro, $\lambda w.$ x had$_w$ [2] churros, $\lambda w.$ x had$_w$ [3] churros, ...$\}$:
$p(w) = 1 \rightarrow p \leq_{Fscale}$ [$\lambda w.$ x had$_w$ [2] churros]]

As in the case of *at least*, the utterance of (29b) in a context where we are trying to find out the maximum number of churros eaten by any linguist [e.g.(31)] would lead to an ignorance inference if the speaker was considered uninformed, or to the inference that at least one linguist had two churros if the speaker was considered authoritative and cooperative. It is only with the latter assumption that an utterance of (29b) would resolve the question, selecting the cell corresponding to [2] in (31):

(31)

$\{w : \exists$x: x is a linguist$_w$ $\wedge$ x has$_w$ [1] churro $\wedge$ $\forall$y: y is a linguist$_w$ $\rightarrow$ y has$_w$ (1] churro $\}$	$\{w : \exists$x: x is a linguist$_w$ $\wedge$ x has$_w$ [2] churros $\wedge$ $\forall$y: y is a linguist$_w$ $\rightarrow$ y has$_w$ (2] churros $\}$	$\{w : \exists$x: x is a linguist$_w$ $\wedge$ x has$_w$ [3] churros $\wedge$ $\forall$y: y is a linguist$_w$ $\rightarrow$ y has$_w$ (3] churros $\}$	.. ..

Given this proposal, the listener's judgement that the speaker is authoritative and cooperative leads to an enrichment that guarantees that the utterance resolves the question in the context. We will build on *authority-driven enrichment* in relation to modals in Section (4).

Why isn't a similar strategy deployed by the listener in the simpler examples we have discussed? Why are ignorance inferences more persistent there? Consider again the effect of *at least* when we are wondering about how many churros Alma had:

(32) a. Alma had at least two$_F$ churros.
b.

~~Alma had [1] churros~~	Alma had [2] churros	Alma had [3] churros
Alma had [4] churros	Alma had [5] churros	Alma had [6] churros

If, upon hearing (32a), the listener were to make the assumption that Alma had [2] churros, the utterance would resolve the question in the context. Why doesn't the listener make this move? A variety of issues arise here, but, crucially, the listener would have no guidance regarding the type of enrichment that the speaker could intend. Given an utterance of (32a), any of the propositions that Alma had [2] churros, or that she had [3] churros, or that she had [4] churros, etc. could be additionally taken on by the listener to arrive at an interpretation that allows the utterance to resolve the question. But only one is correct, and the listener has no way of knowing which. It seems reasonable for the listener to conclude that the speaker is ignorant (as opposed to uncooperative).

3.2 *Disjunction*-flavoured superlatives

In the proposal above, the interpretation of *at least* and *at most* associates with the (ordered) focus-alternatives of the clause in their scope. An alternative approach was outlined by Büring (2008), who proposed that *at least* and *at most* were, in essence, *disjunctive*. From Buring's perspective, the interpretation of the superlatives in (33a) and (34a) was parallel to that of the corresponding disjunctions and, as expected, gave rise to ignorance intuitions associated with the disjuncts:

(33) a. Alma had at least two churros.
b. Alma had exactly two churros or Alma had more than two churros.

(34) a. Alma had at most two churros.
b. Alma had exactly two churros or Alma had less than two churros.

Since Büring (2008), the relation between superlatives and disjunction has been conceptualized in various manners, with proposals for the interpretation of superlatives that often do not directly encode disjunction. I cannot do justice to this complex literature here, which includes neo-Gricean scalar proposals (e.g. Kennedy (2015) Schwarz (2016)), grammaticalized exhaustivity (Buccola & Haida (2018), Mihoc (2019a)), inquisitive approaches (Coppock & Brochhagen (2013), Blok (2019)), and enrichment driven by a *neglect zero* assumption (Aloni & van Ormondt (2023)). In this section I provide a simplified neo-Gricean sketch that derives ignorance inferences in terms of Quantitiy Scalar Implicatures (SIs), with the hope that it give readers a sense of how *at least* and *at most* can be linked to disjunction (independently of focus). With the assumption that *at least* is associated with *exactly* and *more than* alternatives by a *Horn-scale* (see Schwarz (2016) for detailed discussion), an utterance of *at least* would give rise to the SI that the speaker did not believe either of the alternatives. In this type of account, it is the *Horn-scale* alternatives of the superlative that are responsible for the *disjunction*-like effects noted in Büring (2008).

(35) A neo-Gricean-style account of ignorance inferences:
a. Alma had at least two churros.
b. Quality-based inference:
$\Box_{Sp}$ (Alma had at least two churros)
c. Quantity-based SI (modulated by Relevance):
$\neg\Box_{Sp}$ (Alma had exactly two churros) $\wedge$ $\neg\Box_{Sp}$ (Alma had more than two churros)
d. Ignorance: $\Diamond_{Sp}$ (Alma had exactly two churros) $\wedge$ $\Diamond_{Sp}$ (Alma had more than two churros)

Through neo-Gricean reasoning appealing to Horn-scales, the prediction is that a speaker (Sp) who utters (35a) will give rise to the inference that they are certain about the content (35b) but not certain about the scalar alternatives (35c). Putting together (35b) and (35c), both disjuncts are understood to be epistemically accessible to the speaker ($\Diamond_{Sp}$), thus accounting for intuitions of speaker-ignorance. In this manner though neo-Gricean proposals do not (necessarily) model context formally in terms of *questions under discussion* (as in Section 2.4), they are able to derive ignorance inferences through considerations of Quality and Quantity modulated by Relevance on the basis of Horn-scales.

Crucially, neo-Gricean analyses of *at least* and *at most* linked to disjunction have a lot to say about the obviation of ignorance in the scope of universal modals (see a.o.Klinedinst (2007), Büring (2008), Schwarz (2016)). Below I expand on the account sketched above. An utterance of (36a) would give rise to the SI in (36b):

(36) a. Every linguist had at least two$_F$ churros.

b. $\neg\Box_{Sp}$ (Every linguist had exactly two churros) $\wedge$ $\neg\Box_{Sp}$ (Every linguist had more than two churros)

The speaker's belief that every linguist had at least two churros enriched with the SI in (36b) is compatible with the speaker being fully knowledgeable regarding how many churros each linguist had. In the scope of a universal quantifier, the scalar alternatives of superlatives can 'distribute' with respect to the domain of the universal and do not give rise to SI-ignorance inferences. So, while the proposal for *at least* in Section (2.2) accounted for the obviation of ignorance inferences in the scope of *every* on the basis of *authority-driven strengthening*, neo-Gricean accounts simply fail to derive them in this environment (which has been taken as a strong point in their favor). In what follows we will examine patterns of *ignorance-obviation* associated with modals to help us understand the scopes of the two kinds of accounts (Sections 4 and 5).

4 Interactions with modals

In this section I turn to the empirical puzzles noted in Section (1). Section (4.1) investigates existential modals, providing an account of why *can* with *at most* can give rise to an *authoritative reading* but *can* with *at least* cannot. In Section (4.2) I turn to universal modals, and show that we correctly predict that *must* can give rise to *authoritative readings* with both *at most* and *at least*. In Section (4.3) I turn to alternative scope options and explain why they do not give rise to *authoritative readings* and in Section (4.4) I briefly review the perspective from an analysis in terms of scalar implicatures.

4.1 Existentials and *authoritative*-readings

We will begin by addressing our first empirical puzzle, and examine the interaction between superlatives and *can* in terms of Buring's *authoritative reading*. Our starting point will be *at most*, and will concern ourselves only with the scope option in (37) (returning to the other in Section (4.3)):

(37) a. Alma can have at most two$_F$ churros.

b. [at most [can [Alma have two$_F$ churros]]]

As noted in Section 1, (37) can be interpreted as putting an upper bound on Alma's churro allowance. The interpretation would be as in (38a), with the F-scale in (38b):

(38) a. $[\![$(37b)$]\!]$(w) = 1 iff
$\forall$p $\in$ $[\![$can [Alma have two$_F$ churros]$]\!]^{Alt}$: p(w) = 1 $\rightarrow$ p$\leq_{F_{scale}}$ [λw. $\exists$w' $\in$Acc$_w$: Alma has$_{w'}$ [2] churros]

b. [$_{F_{scale}}$ λw.$\exists$w' $\in$Acc$_w$: Alma has$_{w'}$ [1] churros $<$ λw. $\exists$w' $\in$Acc$_w$: Alma has$_{w'}$ [2] churros $<$ λw. $\exists$w' $\in$Acc$_w$: Alma has$_{w'}$ [3] churros $<$ ...]

According to (38a), (37b) will be true in a world *w* iff there is no accessible world where Alma has more than [2] churros (she is not allowed more = the *upper-bound* reading). The interpretation in (38a) does not guarantee that Alma is indeed allowed to have two churros (it just says that Alma is not allowed to have more than two). However, following our reasoning for the case of *every*, we can conclude that the interpretation will be enriched under the assumption that the utterance, made by an authoritative speaker, resolves the question. Consider the toy context in (39), where it is accepted that there is a range in the number of churros permitted to Alma, but there is a question as to what the maximum permitted amount is:

(39)

$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [1] churro $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (1] churro $\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [2] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (2] churros $\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [3] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (3] churros $\}$	... ...

⟦(37b)⟧ would eliminate the third (and subsequent) cells but would not resolve the question. The assumption that the utterance has been made by an authoritative and cooperative speaker, however, would lead to the *authority-driven* enrichment (*Alma is allowed to have [2] churros*), selecting the second cell and eliminating the first, thus resolving the question. Note that enrichment with *Alma is allowed to have [1] churro* would not have led to a resolution of the questions, since both the first and second cell would survive that update.

Let us turn now to the absence of an authority-reading for the case of *can* interacting with *at least*. Consider the *at least*-variant of (37) with wide-scope *can* (40b) and a context where the question is about the minimum permitted amount (41):

(40) a. Alma can have at least two$_F$ churros.
b. [can [at least [Alma have two$_F$ churros]]]

(41)

$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [1] churro $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [1) churro$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [2] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [2) churros$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [3] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [3) churros$\}$	... ...

Learning that the minimum was [2] would resolve the question in the context (the second cell would be selected). However, this will not be the update brought about by an utterance of (40b). Consider the predicted interpretation in (42a) given the F-scale in (42b):

(42) a. $[\![(40b)]\!](w) = 1$ iff $\exists w' \in$Acc$_w$: $\exists p \in [\![$Alma have two$_F$ churros$]\!]^{Alt}$: $p(w') = 1$ $\wedge\, p \geq_{F_{scale}}$ [λw.Alma has$_w$ [2] churros]
b. [$_{F_{scale}}$ λw.Alma has$_w$ [1] churro $<$ λw.Alma has$_w$ [2] churros $<$...]

Given this interpretation, an utterance of (40b) would not eliminate ANY of the cells in (41). It would obviously not eliminate the option that [2] churros are the minimum, nor [3] churros nor any higher amount. But it would also fail to eliminate the cell corresponding to [1] as the minimum, since in that cell there are accessible worlds where Alma has more than one churros. There is no strengthening strategy that could remedy this situation, so the prediction is that given such a question, an utterance of (40b) will be infelicitous. Alternating the relative scope of the existential superlative and existential modal will not make any difference:

(43) a. [at least [can [Alma have two$_F$ churros]]]
b. $[\![(43a)]\!](w) = 1$ iff $\exists p \in [\![$can [Alma have two$_F$ churros]$]\!]^{Alt}$: $p(w) = 1$ $\wedge\, p \geq_{F_{scale}}$ [λw.$\exists$w'$\in$Acc$_w$: Alma has$_{w'}$ [2] churros]

c. $[_{F_{scale}}\ \lambda$w.$\exists$w'$\in$Acc$_w$: Alma has$_{w'}$ [1] churros $<\lambda$w.$\exists$w'$\in$Acc$_w$: Alma has$_{w'}$ [2] churros $<\lambda$w.$\exists$w'$\in$Acc$_w$: Alma has$_{w'}$ [3] churros $<$ ...]

Parallel to (40b), an utterance of (43a) would also fail to resolve the question in (41). It is thus predicted that, contrary to *at most*, *at least* interacting with *can* will fail to give rise to an authoritative-reading.

What if it was presupposed that Alma was required/permitted to have a specific amount of churros? In that case, both *at least* and *at most* as discussed above would be informative, but lead to speaker ignorance inferences:

(44)

$\{w : \forall w' : w' \in Acc_w \to$ Alma has$_{w'}$ [1] churro$\}$	$\{w : \forall w' : w' \in Acc_w \to$ Alma has$_{w'}$ [2] churros$\}$	$\{w : \forall w' : w' \in Acc_w \to$ Alma has$_{w'}$ [3] churros$\}$	.. ..

An utterance of (37b), for example, would discard the third cell in (44) (and higher amounts), but both the first and second cell would survive the update. (40b), on the other hand, would discard the first cell, but the others would remain (I repeat the relevant interpretations below):

(45) a. $[\![$(37b)$]\!]$(w) = 1 iff
$\forall$p $\in [\![$can [Alma have two$_F$ churros]$]\!]^{Alt}$: p(w) = 1 $\to$ p$\leq_{F_{scale}}$ [λw. $\exists$w' $\in$Acc$_w$: Alma has$_{w'}$ [2] churros]

b. $[\![$(40b)$]\!](w) = 1$ iff $\exists w' \in$Acc$_w$: $\exists p \in [\![$Alma have two$_F$ churros$]\!]^{Alt}$: $p(w') = 1 \land p \geq_{F_{scale}}$ [λw.Alma has$_w$ [2] churros]

Again, no strengthening strategy would allow the listener to bypass this problem.[3]

4.2 Authoritative readings of universals

We turn now to the case of *must*, which allows *authoritative readings* with both *at least* and *at most*. Consider *at least* in (46a), which can be interpreted as the clarification of the lower bound on Alma's churros-eating obligations by an authoritative speaker:

(46) a. Alma must have at least two$_F$ churros.

b. [must [at least [Alma has two$_F$ churros]]]

The structure in (46b) will have the denotation in (47a) and the associated F-scale will be as in (47b):

(47) a. $[\![$46b$]\!]$(w) = 1 iff
$\forall$w'$\in$Acc$_w$: $\exists$p$\in [\![$Alma has two$_F$ churros$]\!]^{Alt}$:p(w') = 1
$\land$ p $\geq_{F_{scale}}$ [λw.Alma has$_w$ [2] churros]

b. $[_{F_{scale}}\lambda$w.Alma has$_w$ [1] churro $<\lambda$w.Alma has$_w$ [2] churros $<$ ...]

Consider again a context in which there is a range of options for Alma's permitted churro-eating, and we are now wondering about the lower bound:

(48)

$\{w : \exists w' : w' \in Acc_w \land$ Alma has$_{w'}$ [1] churro $\land$ $\forall w" : w" \in Acc_w \to$ Alma has$_{w"}$ [1) churro$\}$	$\{w : \exists w' : w' \in Acc_w \land$ Alma has$_{w'}$ [2] churros $\land$ $\forall w" : w" \in Acc_w \to$ Alma has$_{w"}$ [2) churros$\}$	$\{w : \exists w' : w' \in Acc_w \land$ Alma has$_{w'}$ [3] churros $\land$ $\forall w" : w" \in Acc_w \to$ Alma has$_{w"}$ [3) churros$\}$	.. ..

3 See Section (4.3) for a discussion of interpretations associated with speaker ignorance.

Updating the context with (46b) will discard the first cell in (48), but in principle the others remain 'live', which means that the question is not resolved. Given our earlier reasoning about enrichments triggered by the assumption of speaker cooperation and authority, the prediction is that the interpretation of (46b) will be enriched with the proposition that there is a permitted world in which Alma has two churros (*authority-driven* enrichment). The upshot is that the second cell in (48) is selected and the question about Alma's minimum obligations is resolved.

Consider now *at most* in (49a), interpreted as in (49b) and with the scale in (49c):

(49) a. [must [at most [Alma has two$_F$ churros]]]

b. $[\![$49a$]\!]$(w) = 1 iff $\forall$w'$\in$Acc$_w$:$\forall$p$\in$ $[\![$Alma has two$_F$ churros$]\!]^{Alt}$:$p(w') = 1 \rightarrow$ $p \leq_{F_{scale}}$ [λw.Alma has$_w$ [2] churros].

c. [$_{F_{scale}}$ λw.Alma has$_w$ [1] churro < λw.Alma has$_w$ [2] churros < ...]

The truth of $[\![$(49b)$]\!]$ in a world w requires that no alternatives in $[\![$Alma has two$_F$ churros$]\!]^{Alt}$ higher than [λw.Alma has$_w$ [2] churro] be true in any worlds accessible to w. Given a context with a question about the upper bound on permitted options (50), the assertion would eliminate all alternatives corresponding to [3] or higher:

(50)

$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [1] churro $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (1] churro$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [2] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (2] churros$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [3] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (3] churros$\}$	... .

Reasoning in a manner parallel to the case of *at least* we can interpret an utterance of (49a) by a cooperative and authoritative speaker as leading to strengthening that resolves the question (Alma is indeed allowed to have [2] churros). With this enrichment, only the second cell in (50) would survive an update with (49a).

The upshot is that wide-scope *must* in combination with *at least* and *at most* allows an authoritative speaker to address questions about lower or upper bounds on a range of permitted options. As we will see in the next section, when the superlative takes scope over *must*, variation in the range of permitted/required options is disallowed and the utterance fails to address such questions.

4.3 Some alternative scope options

Our discussion in Sections (4.1) and (4.2) focused on authoritative readings available to particular scope relations between superlatives and modals. In this section we will investigate other scope options, beginning with the case of *can* interacting with *at most*. Consider an interpretation of (37) with the alternative scope relations in (51a) (compare with (37b)):

(51) a. [can [at most [Alma have two$_F$ churros]]]

b. $[\![$(51a)$]\!]$(w) = 1 iff
$\exists$w' $\in$Acc$_w$: $\forall$p $\in$ $[\![$Alma has two$_F$ churros$]\!]^{Alt}$: p(w') = 1 $\rightarrow$ p$\leq_{F_{scale}}$ [λw. Alma has$_w$ [2] churros]

Given this scope configuration, the utterance would not resolve a question about an upper bound on permitted churros. To see this, recall what the context would look like in such a case:

(52)

$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [1] churro $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (1] churro $\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [2] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (2] churros $\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [3] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ (3] churros $\}$	.. ..

The update brought about by (51a) would not discard any of the cells in (52). In all of them it is allowed for Alma to eat [2] churros or less. And in this case, there is no obvious strengthening strategy that could help, e.g. strengthening with the proposition that $\exists w' \in Acc_w$*: Alma has*$_{w'}$ *[2] churros* will not have as outcome the resolution of the question, since the second and third/subsequent cells would survive the update.

Note that while (51a) could not resolve a question about the upper bound, it could resolve a *yes/no* question as to whether Alma is allowed to stop at two churros. Given such a question, the context would consist of basically two cells, one that collapsed all alternatives that only had accessible worlds in which Alma had three churros or more, and another that collapsed all alternatives that had at least one accessible world where Alma had two churros or less. Here is an illustration of what such a context could look like:

(53)

$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ (2] churros	$\{w : \forall w' : w' \in Acc_w \rightarrow$ Alma has$_{w'}$ [3) churros

In such a context, an utterance of (51a) by an authoritative speaker would make clear that it is not necessary for Alma to get to [3] churros. An utterance of (51a) would eliminate the second cell in (53) and resolve the question. [4] [5]

Let us now consider again the case of *must*. An utterance of (46a) with wide-scope *at least* would not resolve a question about the lower bound (repeated below):

(54)

$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [1] churro $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [1) churro$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [2] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [2) churros$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [3] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [3) churros$\}$	... ...

The structure would be as in (55a) (compare with (46b)) and the interpretation as in (55b), with the relevant scale in (55c):

(55) a. [at least [must [Alma has two$_F$ churros]]]

b. $[\![$(55a)$]\!]$(w) = 1 iff
$\exists$p$\in$ $[\![$must [Alma has two$_F$ churros]$]\!]^{Alt}$: p(w) = 1 $\wedge$
p $\geq_{F_{scale}}$ [λw.$\forall$w'$\in$Acc$_w$: Alma has$_{w'}$ [2] churros]

c. [$_{F_{scale}}$ λw.$\forall$w'$\in$Acc$_w$: Alma has$_{w'}$ [1] churro < λw.$\forall$w'$\in$Acc$_w$: Alma has$_{w'}$ [2] churros< λw.$\forall$w'$\in$Acc$_w$: Alma has$_{w'}$ [3] churros ...]

An utterance of (55a) will be true iff Alma is obliged to have a particular amount of churros, with the amount equal to [2] or higher. The utterance will only be felicitous in a context in which it

4 The two scopes are not independent. If the maximum permitted amount of churros for Alma is [2], then there will be a permitted world where Alma has [2] churros. The reverse entailment does not hold: Alma may be allowed to stop at [2] churros without being forbidden from having more.

5 I will not have anything to say regarding the relation between intonation and the availability of this reading. The matter remains for future (urgent) research.

is presupposed that there is an obligation about a specific amount, with a question as to the exact amount (e.g. (44)). Given that (55a) does not actually settle the amount, the prediction is that in a context in which the utterance is felicitous, it will fail to fully resolve the question, leading to ignorance inferences. The same will occur with *at most* scoping over *must*:

(56) a. [at most [must [Alma has two$_F$ churros]]]

b. $[\![$(56a)$]\!]$(w) = 1 iff
$\forall$p $\in$ $[\![$must [Alma has two$_F$ churros]$]\!]^{Alt}$: p(w) = 1 $\rightarrow$ p $\leq_{F_{scale}}$ [λw. $\forall$w' $\in$Acc$_w$: Alma has$_{w'}$ [2] churros]

c. [$_{F_{scale}}$ λw.$\forall$w'$\in$Acc$_w$: Alma has$_{w'}$ [1] churro $<$ λw.$\forall$w'$\in$Acc$_w$: Alma has$_{w'}$ [2] churros $<$ λw.$\forall$w'$\in$Acc$_w$: Alma has$_{w'}$ [3] churros $<$ ...]

Given the assumption that there is variation in the number of permitted churros and a question about the upper bound (50), an utterance of (56a) would be infelicitous. And given the assumption that there is no variation and a question about the exact amount, an utterance of (56a) would fail to resolve the question. The result is that in combination with *at least* and *at most*, only wide-scope *must* allows an authoritative speaker to resolve a question about lower or upper bounds.

4.4 A view from disjunction

The enrichments we have noted in the interaction of *at least* and *at most* with quantifiers bear similarity to so-called *distributivy*-inferences often associated with disjunction. Distributivity-inferences have been noted in the literature in relation to both existential and universal quantifiers:

(57) a. You may have cake or icecream.
You may have cake and you may have ice-cream. (*distributivy*-inference)

b. You must have cake or icecream.
You may have cake and you may have ice-cream. (*distributivy*-inference)

c. Everyone had cake or icecream.
Some people had cake and some people had icecream. (*distributivy*-inference)

It has been observed that when disjunction takes scope under an existential (deontic) modal, it gives rise to the inference that there is *free choice* amongst the disjuncts (57a), and the same thing happens when it takes scope under a univeral modal (57b). The pattern is reproduced for universals ranging over entities (57c). The explanation for these inferences varies significantly depending on the account. Here I simply note that establishing links between superlatives and disjunctions potentially provides insights into the interpretation of superlatives interacting with modals. (58b) spells out the distributivity-inference that would be predicted as a *free-choice* effect by an analysis that linked *at most* to a disjunction of *exactly* and *less than*:

(58) a. Alma may have at most two churros.

b. Alma may have exactly two churros and Alma may have less than two churros.

How would such an account compare to the *authority-driven strengthening* proposal in Section (4.1)? The accounts would be similar in predicting that both [2] churros and (2) churros are allowed. However, contrary to the proposal in Section (4.1), an analysis in terms of free-choice would not (immediately) predict that [2] churros is an *upper-bound* on what is allowed. Indeed, proposals defending *free-choice*-style analysis of (58b) have often noted that additional moves would be needed to derive the authoritative upper-bound interpretation. Here are some examples and remarks from Kennedy (2015), who proposed a neo-Gricean analysis deriving distributivity in (58a) as a quantity SI from the mechanisms that generate free-choice in examples like (57a).

(59) a. You are allowed to register for at most three classes.
b. You are allowed to register for exactly three classes and you are allowed to register for less than three classes. (*distributivity/ free choice* inference)
c. "If the question under discussion is what the rules say about the number of classes that can be taken, if the knowledgeable speaker tells us that the rules permit registration in zero to three (the conjunction of the truth conditions plus the quantity implicatures), then we should assume that this is all that the rules allow." Kennedy (2015).

As Kennedy notes, the 'closure' effect that accounts for the upper bound reading is not linked to the scalar implicatures triggered by the superlative but must be derived through additional pragmatic reasoning. The proposal developed in Section (4.1), in contrast, encodes pragmatic reasoning by building on questions in the discourse directly, and is able to account for the authoritative reading without characterizing *exactly...* and *less than....* as (Horn-scale) alternatives to *at most*.

An additional difference between disjunction-based accounts and the proposal in Section (4.1) is apparent in the predictions made in relation to existential modals interacting with *at least*. In disjunction-inspired accounts of superlatives, *at least* and *at most* are essentially symmetric, differing only with respect to one of the disjuncts involved (*more than* vs. *less than*). Given this symmetry, the absence of an authoritative reading for *at least* interacting with existentials is hard to explain. Consider an *at least* version of Kennedy's example (59a):

(60) a. You are allowed to register for at least three classes.
b. You are allowed to register for exactly three classes and you are allowed to register for more than three classes. (expected *distributivity/ free choice* inference).

In accordance with Büring (2008), I take (60a) to lack an authoritative reading: a speaker cannot utter (60a) to set a lowest-bound on the permitted class number. Yet, following the reasoning outlined for (59a), we would predict that distributivity inferences (60b) plus additional pragmatic reasoning should lead to the inference that the disjuncts in (60b) spell out the only allowed alternatives, i.e. a lowest-bound reading. The absence of such a reading for (60a) remains unexplained in an account that links both *at least* and *at most* to disjunction in a manner that is symmetrical. The proposal in Section(4.1) fares better in accounting for the difference between existential modals interacting with *at most* vs. *at least*.

5 Extending the analysis and concluding remarks

In this section I offer some concluding remarks speculating beyond the case of *at least* and *at most*. What about their comparative counterparts *less than* and *more than*? Building on Fox & Hackl (2006), Mayr (2013) observed that differences between existential and universal modals interacting with *at least* carry over to *more than*. Following Mayr (2013), the point is illustrated with the examples below:

(61) a. Doctor: Alma must have more than two churros.

b. $\rightsquigarrow \neg$ Alma must have more than three churros.

(62) a. Doctor: Alma can have more than two churros.
b. $\not\rightsquigarrow \neg$ Alma can have more than three churros.

Like *at least*, *more than* interacting with a universal modal (61a) can be understood as setting a lower bound. Consider again a context with a question about lower bound on required churros:

(63)

$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [2] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [2) churros$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [3] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [3) churros$\}$	$\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [4] churros $\wedge$ $\forall w" : w" \in Acc_w \rightarrow$ Alma has$_{w"}$ [4) churros$\}$	... ...

With the assumption that *more than* does not associated with focus (see e.g. Fox & Hackl (2006), Coppock & Brochhagen (2013)), let us grant that an utterance of (61a) would be interpreted as (64b):

(64) a. [Must [Alma has more than two churros]]
b. $[\![$(61a)$]\!]$(w) = 1 iff ∀w'∈Acc$_w$: max{n: Alma has$_{w'}$ n churros} > 2

The prediction is that an utterance of (61a) would lead to an update of (63) that would discard the cell corresponding to [2] (the 'first' cell). At this stage, the prediction is that the utterance would not fully resolve the question, and lead to an ignorance inference. However, as in the case of *at least*, an addressee who make the assumption that the speaker is cooperative and authoritative could reason further. In order to preserve such assumptions about the speaker, the addressee could conclude that the speaker intends the addressee to take on a commitment to the proposition that Alma is allowed to have three churros ($\exists w' : w' \in Acc_w$& Alma has$_{w'}$ [3] churros). With this enrichment, an utterance of (61a) would resolve the question in (63). Taking on this additional commitment amounts to accepting that the minimum permitted number of churros is not higher than 3 (i.e. the inference in (61a): *Alma is not obliged to eat more than 3 churros*). The pattern in (61a) thus follows simply from allowing the pragmatic mechanism of *authority-driven* strenghtening familiar from the case of *at least* to also play out in the example with *more than*. Since there is nothing *at least*-specific about the reasoning that leads to strengthening in our earlier examples, nothing blocks it in the case of *more than*. The conclusion is that a universal modal can set a lower bound on what is allowed in interaction both with *at least* and *more than*.

Consider now the existential case in (62a), with the interpretation in (65b):

(65) a. [Can [Alma has more than two churros]]
b. $[\![$(65a)$]\!]$(w) = 1 iff ∃w'∈Acc$_w$: max{n: Alma has$_{w'}$ n churros} > 2

Could an utterance of this sentence resolve the question in (63)? Recall that an existential modal interacting with *at least* was not able to resolve the question about the lowest bound (Section (4.1)). The same pattern emerges for *more than*. An utterance of (62a) would not discard any of the cells in (63) and thus would not even address such a question. There is no univocal strengthening strategy that would allow the question in (63) to be resolved, and no inferences are made regarding whether or not Alma is forbidden from eating more than three churros. The only contribution that (62a) could make is to clarify that eating more than two churros is not deontically ruled out for Alma. The utterance would thus resolve a Yes/No question without giving rise to ignorance inferences:

(66) | $\{w : \exists w' : w' \in Acc_w \wedge$ Alma has$_{w'}$ [3) churros | $\{w : \forall w' : w' \in Acc_w \rightarrow$ Alma has$_{w'}$ (2] churros |
|---|---|

The proposal above can serve as a sketch of how to account for (some) similarities between superlatives and comparatives interacting with modals. As noted in Mayr (2013), similarities in the enrichment patterns observed with superlatives and comparative numerals suggests that they have a common root, favoring an account that can maintain a unified perspective across both sets of phenomena. As also pointed out by Mayr (2013), the proposals in the literature often focus on one type of modified numeral only, missing the generalizations across the domains. A notable exception is Mihoc (2019a) and Mihoc (2019b), that provide a detailed analysis of superlative and comparative numerals building on comparisons with disjunctions and indefinites. Mihoc's main focus is on the relation between ignorance inferences and polarity, a topic I have not addressed and remains for future research.

To conclude: In this paper I have presented a proposal that leads to pragmatic enrichment on the basis of reasoning about properties of the speaker (enrichment as a *sanity check*). The underlying hypothesis is that the speaker's goal is to resolve a question in the discourse, and the account requires taking on commitments about the information tracked by context. Coupled with a focus-based semantics for superlatives, the upside of the proposal has been insights into differences in how modals interact with superlative modifiers that have proven challenging for other accounts

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