

Staged updates*

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Abstract Dynamic semantics standardly treats negation as externally static. This is inconsistent with anaphora to doubly-negated indefinites, and Partee disjunctions. I propose a technique for *lifting* dynamic systems into richer meaning spaces that support dynamic negation. I argue that extant theories of dynamic negation instantiate this technique in various ways, prove that all such lifted systems are isomorphic, and identify a minimal/canonical enrichment of Dynamic Predicate Logic. I relate this perspective on dynamic negation to recent work on ‘bounded’ meaning (Mandelkern 2022). Finally, I consider how such lifted interpretations with dynamic negations can be extended further, into richer meaning spaces that support exceptional scope of indefinites (Charlow 2025).

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1 Dynamic semantics and double negation

Dynamic theories of anaphora offer an appealing explanation of how an indefinite can antecede a pronoun it doesn’t scope over.¹ In (2), a natural compositional translation of (1) into predicate logic (PL), the pronoun occurs outside the syntactic scope of $\exists x$. Given the usual static interpretation of PL, (2) does not represent the reading of interest, in which the two sentences’ subjects are covalued.

- (1) A^x linguist walked in the park. She _{x} smiled.
(2) $\exists x(Lx \wedge Wx) \wedge Sx$

Dynamic semantics adopts a fundamentally enriched perspective on meaning, which allows indefinites to extend their binding scope rightward, such that $\exists x$ semantically binds the syntactically free occurrence of x in (2), as desired.

In static frameworks, a sentence ϕ ’s interpretation at an index i (an assignment, or world-assignment pair) is a truth value $\llbracket \phi \rrbracket^i \in \{\text{T}, \text{F}\}$. Equivalently, ϕ ’s static meaning *simpliciter* is the set of indices where ϕ is true: $\llbracket \phi \rrbracket := \{i : i \mid \llbracket \phi \rrbracket^i\}$, type $i \rightarrow \text{t}$.

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¹ Kamp 1981; Heim 1982; Rooth 1987; Groenendijk & Stokhof 1991a; Dekker 1993; Muskens 1996.

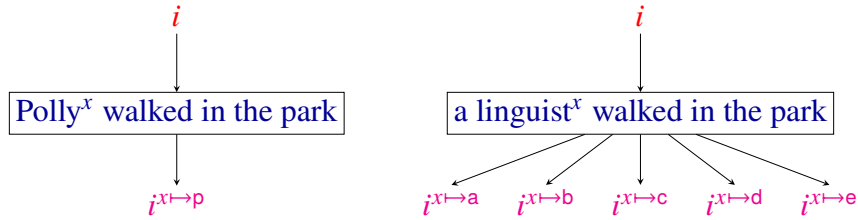


Figure 1 Introducing discourse referents by updating indices.

In dynamic theories, by contrast, sentence meanings encode **updates** on a body of information. The basic idea is depicted in Figure 1. Intuitively, a utterance of *Polly^x walked in the park* makes Polly salient as a possible discourse referent for subsequent pronouns. This is formally cashed out by treating the sentence as characterizing an update that reads in an input index i and outputs a modified index $i^{x \rightarrow p}$ in which x now points to Polly. Sentences with indefinites work analogously, but because there may be more than witness for any indefinite (e.g., if multiple linguists walked in the park), the updates they characterize may be nondeterministic (yielding more than one output).

Probably the minimal system that makes this all go is Dynamic Predicate Logic (DPL, Groenendijk & Stokhof 1991a). DPL views sentence meanings as relations on indices, type $i \rightarrow i \rightarrow t$ (which is plainly richer than the static type $i \rightarrow t$):

Definition 1 (Dynamic Predicate Logic).

$$\begin{aligned}
 i[Fx_1 \dots x_n] &:= \{i \mid \llbracket Fx_1 \dots x_n \rrbracket^i\} & i[\phi \wedge \psi] &:= \{k \in j[\psi] \mid j \in i[\phi]\} \\
 i[\exists x] &:= \{i^{x \rightarrow d} \mid d \in D\} & i[\neg \phi] &:= \{i \mid i[\phi] = \emptyset\}
 \end{aligned}$$

Definition 2 (DPL truth and falsity).

$$\text{True}(\phi, i) := i[\phi] \neq \emptyset \qquad \text{False}(\phi, i) := i[\phi] = \emptyset$$

In line with Figure 1, we use an iconic postfix notation $i[\phi]$ to represent the update $[\phi]$ applied to the input i . Interpretations for atomic ϕ return the input i unchanged, conditional on ϕ 's truth at i . Hence, truth at an input i corresponds to returning at least one output (success), and falsity to returning no outputs (failure) (Definition 2).

Existential quantifiers introduce discourse referents (the dynamic setup allows us to treat $\exists x$ as an independent formula; $\exists x(\phi)$ may be understood as abbreviating $\exists x \wedge \phi$). Conjunction sequences updates via relation composition, passing the outputs j of the first conjunct as inputs to the second. And negation returns the input index i unchanged, conditional on the prejacent being false at i .

Putting all this together, (2) receives the following DPL interpretation, which covalues the linguist who walked in the park and the smiler, exactly as required:

$$(3) \quad \lambda i. \{ i^{x \rightarrow d} \mid d \in D, \llbracket Lx \rrbracket^{i^{x \rightarrow d}}, \llbracket Wx \rrbracket^{i^{x \rightarrow d}}, \llbracket Sx \rrbracket^{i^{x \rightarrow d}} \}$$

As lovely as this all is, there's a problem. DPL negation is **externally static**. $\neg\phi$ naturally insists on the falsity of ϕ . But because DPL falsity at an input i corresponds to lacking any outputs (failure), $\neg\phi$, when true, can at best return only the unmodified input i . It follows that $\neg\phi$ never introduces discourse referents, for any ϕ whatsoever.

Though externally static negation works well for examples like (4), where a single negation disrupts anaphora, it's problematic elsewhere. Adding a second negation as in (5), or the less stilted denial in (6B), renders anaphora acceptable. Likewise for the 'Partee disjunction' (7), where anaphora is seamless, although the indefinite occurs under a single negation in the left disjunct.²

- (4) Mary doesn't have a^x child. *She_x's at boarding school.
 (5) It isn't true that Mary doesn't have a^x child. She_x's just at boarding school.
 (6) A: Mary doesn't have a^x child. B: That's not true; she_x's just at boarding school.
 (7) Either Mary doesn't have a^x child, or she_x's at boarding school.

Schematically, examples (5) and (6B) correspond to logical forms like (8) and example (7) to (9), where we exploit the classical equivalence between $\phi \vee \psi$ and $\neg(\neg\phi \wedge \neg\psi)$. Because dynamic negation is externally static, $\neg\neg\exists x(\phi)$ is not equivalent to $\exists x(\phi)$: the latter may introduce discourse referents, but the former never does. DPL thus incorrectly predicts that anaphora to a negated indefinite should never be possible.

- (8) $\neg\neg\exists x(\phi) \wedge \psi$
 (9) $\neg(\neg\neg\exists x(\phi) \wedge \neg\psi)$

Intuitively, what's needed is an externally dynamic negation, and in particular one that enjoys double negation elimination (DNE), that is, whereby $\neg\neg\phi \equiv \phi$ in general. Then (8) $\equiv \exists x(\phi) \wedge \psi$ and (9) $\equiv \neg(\exists x(\phi) \wedge \neg\psi)$, both of which allow $\exists x$ to bind into ψ .

2 Lifting to DNE (abstractly)

The importance and problematic nature of dynamic negation were appreciated quickly (e.g., Groenendijk & Stokhof 1990, 1991a; Dekker 1993). Krahmer & Muskens 1995 gives an early bilateral extension of DPL that supports DNE. Recently, there has been a flurry of work on dynamic negation (e.g., Gotham 2019; Hofmann 2019, 2025; Mandelkern 2022; Elliott 2020). We'll survey much of this work as we go. For now, two points. First, although many theories have been put forward, there is little consensus on

² The examples are borrowed from Mandelkern (2022). As he points out, parallel examples that replace *Mary doesn't have a child* with the truth-conditionally equivalent *Mary isn't a parent* are degraded or unacceptable across the board. This strongly suggests that the acceptability of (5), (6) and (7) is due to a discourse referent introduced by the indefinite *a child*, which manages to survive under negation.

the best approach, and little understanding of how various theories relate to each other. Next, many of these approaches turn out to be **enrichments** of an underlying dynamic semantics (much as dynamic theories enrich static ones), provisioning some dynamic **substrate** with representational resources that allow dynamic negation to be defined.

One of my main contentions in this paper is that it is fruitful to recast the problem of dynamic negation as a question of how to go about enriching a dynamic system into one with DNE. In this section, we first pose the question *abstractly*, that is, by considering the minimal amount of structure we would have to impose to get what we're after.

We begin with a substrate dynamic type δ and a compositional interpretation $[\cdot] : \mathcal{L} \rightarrow \delta$ for a (let's say) first-order language \mathcal{L} validating characteristic dynamic equalities like $\exists x(\phi) \wedge \psi \equiv \exists x(\phi \wedge \psi)$. Given compositionality of $[\cdot]$, we have $\neg_\delta : \delta \rightarrow \delta$, whereby $[\neg\phi] := \neg_\delta[\phi]$, and $\wedge_\delta : \delta \rightarrow \delta \rightarrow \delta$, whereby $[\phi \wedge \psi] := [\phi] \wedge_\delta [\psi]$. To illustrate, we might take a concrete δ to be $\text{DPL} ::= i \rightarrow i \rightarrow t$. Then:

$$\neg_{\text{DPL}} m := \lambda i. \{i \mid m(i) = \emptyset\} \quad l \wedge_{\text{DPL}} r := \lambda i. \{k \in r(j) \mid j \in l(i)\}$$

Given δ without DNE, what would it mean to have an enriched Δ with DNE? We should have mappings $\uparrow : \delta \rightarrow \Delta$ and $\downarrow : \Delta \rightarrow \delta$ for lifting into and lowering out of Δ , together with $\sim : \Delta \rightarrow \Delta$ representing an involutive (self-cancelling) negation in Δ . Because Δ is richer than δ , we require $m^{\uparrow\downarrow} = m$ (that is, \uparrow **embeds** δ meanings faithfully in Δ). Because \sim is **involutive**, we require $\sim\sim M = M$. And because \sim is Δ 's version of δ 's **negation** modulo DNE, we require $(\sim m^{\uparrow})^{\downarrow} = \neg_\delta m$.³ These points are summarized in Definition 3, where we also name the laws regulating the interplay of \uparrow , \downarrow , and \sim .

Definition 3 (Signatures and laws). Fix types δ (a ‘substrate’ dynamic type) and Δ (a ‘lifted’ dynamic type). Given a language \mathcal{L} with wff $\phi ::= \text{Atom} \mid \exists x \mid \phi \wedge \phi \mid \neg\phi$:

$$\begin{array}{lll} [\cdot] : \mathcal{L} \rightarrow \delta & \uparrow : \delta \rightarrow \Delta & m^{\uparrow\downarrow} = m \quad \mathbf{Emb} \\ \neg_\delta : \delta \rightarrow \delta & \downarrow : \Delta \rightarrow \delta & \sim\sim M = M \quad \mathbf{Inv} \\ \wedge_\delta : \delta \rightarrow \delta \rightarrow \delta & \sim : \Delta \rightarrow \Delta & (\sim m^{\uparrow})^{\downarrow} = \neg_\delta m \quad \mathbf{Neg} \end{array}$$

Given $[\cdot]$ and these mappings, we have everything we need to characterize a lifted interpretation function $\langle \cdot \rangle : \mathcal{L} \rightarrow \Delta$. On the left-hand side of Definition 4, we set out an interpretation for δ , as described above. $\text{Prim}(\phi)$ is a placeholder for whatever primitive meaning $[\cdot]$ gives atomic/ $\exists x \phi$. Then $\langle \cdot \rangle$ is defined in a type-directed way, indeed, in the only *possible* way given the resources at hand and the requirement of DNE. Interpretations for atomic and (bare) existential ϕ are simple lifts of the corresponding δ meanings. Conjunction works recursively; we interpret and lower the

³ We should *not* require that $M^{\downarrow\uparrow} = M$, nor that $(\sim M)^{\downarrow} = \neg_\delta M^{\downarrow}$. The first inappropriately strengthens the relationship between δ and Δ to one of isomorphism, but Δ will in general be properly richer than δ . The second contradicts involutivity of \sim , since it entails that $(\sim\sim M)^{\downarrow} = \neg_\delta \neg_\delta M^{\downarrow}$.

individual conjuncts, sequence the lowered meanings with \wedge_δ , and lift the result back into Δ .⁴ Finally, negation in \mathcal{L} is mapped, naturally, to the involutive Δ negation \sim .

Definition 4 (Interpretations). Let $[\cdot] : \mathcal{L} \rightarrow \delta$ and $\langle \cdot \rangle : \mathcal{L} \rightarrow \Delta$ be defined by:

$$\begin{aligned} [\phi] &:= \text{Prim}(\phi) \quad (\phi \text{ atomic or } \exists x) & \langle \phi \rangle &:= [\phi]^\uparrow \quad (\phi \text{ atomic or } \exists x) \\ [\phi \wedge \psi] &:= [\phi] \wedge_\delta [\psi] & \langle \phi \wedge \psi \rangle &:= (\langle \phi \rangle^\downarrow \wedge_\delta \langle \psi \rangle^\downarrow)^\uparrow \\ [\neg \phi] &:= \neg_\delta [\phi] & \langle \neg \phi \rangle &:= \sim \langle \phi \rangle \end{aligned}$$

An immediate consequence is that any such lifted interpretation $\langle \cdot \rangle$ validates DNE:

Fact 1 (Lifted interpretations validate DNE). Given $\langle \cdot \rangle$ respecting Definitions 3 and 4:

- i. $\langle \neg \neg \phi \rangle = \sim \langle \neg \phi \rangle = \sim \sim \langle \phi \rangle = \langle \phi \rangle$ by $\langle \cdot \rangle$ and Inv.
- ii. $\langle \dots \neg \neg \phi \dots \rangle = \langle \dots \phi \dots \rangle$ by (i) and compositionality of $\langle \cdot \rangle$.

With a suitable semantics for disjunction, Partee disjunctions like (7) follow as well. As above, we might define $\phi \vee \psi := \neg(\neg \phi \wedge \neg \psi)$, then calculate as follows:

$$\begin{aligned} \langle \neg \exists x(\phi) \vee \psi \rangle &= \langle \neg(\neg \neg \exists x(\phi) \wedge \neg \psi) \rangle && \text{(def. of } \vee) \\ &= \langle \neg(\exists x(\phi) \wedge \neg \psi) \rangle && \text{(by Fact 1.ii)} \end{aligned}$$

For typical substrates (ones like DPL in relevant respects), this meaning is *strong*. For (7), this means that all of Mary's children (if she has any) are at boarding school. Because disjunction is defined in terms of an outer negation, bare (i.e., non-negated) disjunctions are also predicted to be externally static. However, weak and externally dynamic meanings for disjunction are easy to define, and choosing among them is largely orthogonal to the concerns of this paper. See Section 4 for discussion.

Lifted interpretations $\langle \cdot \rangle$ that conform to Definitions 3 and 4 are fundamentally **conservative** over $[\cdot]$. Informally, $\langle \cdot \rangle$ is 'tight': it abstractly characterizes how to move from a substrate dynamic semantics to one that adds DNE *and nothing else*, leaving all other properties of the substrate dynamics intact. Fact 2 offers two views on this claim: if ϕ is free of negations, $\langle \phi \rangle$ is simply the lift of the substrate interpretation $[\phi]$; if ϕ is free of double-negations, $\langle \phi \rangle^\downarrow$ is simply $[\phi]$. These facts follow from inductions on \mathcal{L} . Proofs are in Appendix A. Truth for $\langle \cdot \rangle$ also piggybacks on the substrate (Definition 5).

Fact 2 (Lifted interpretations are conservative over $[\cdot]$).

- i. For \neg -free ϕ : $\langle \phi \rangle = [\phi]^\uparrow$
- ii. For $\neg \neg$ -free ϕ : $\langle \phi \rangle^\downarrow = [\phi]$

⁴ $\langle \phi \wedge \psi \rangle$ cannot be defined as $[\phi \wedge \psi]^\uparrow$. This wrongly interprets ϕ and ψ using $[\cdot]$, which lacks DNE.

Definition 5 (Truth for lifted interpretations). Given a function $f : (\delta \times i) \rightarrow \mathfrak{t}$ such that for any ϕ , $\text{True}_{[\cdot]}(\phi, i) = f([\phi], i)$, $\text{True}_{\langle \cdot \rangle}(\phi, i) := f(\langle \phi \rangle^\downarrow, i)$.

Any lifted interpretation hence accounts for examples (5), (6), and (7), and moreover does so tidily, i.e., without disturbing whatever fundamental properties make up the substrate dynamics, modulo DNE.

Last, we consider the impossibility of anaphora to a singly-negated indefinite, as in (4). We first calculate an abstract lifted interpretation for the first conjunct, as follows:

$$\begin{aligned} \langle \neg \exists x(\phi) \rangle &= \sim \langle \exists x(\phi) \rangle && \text{(by } \langle \cdot \rangle \text{)} \\ &= \sim \langle \exists x \wedge \phi \rangle && \text{(def. of } \exists x(\phi) \text{)} \\ &= \sim (\langle \exists x \rangle^\downarrow \wedge_\delta \langle \phi \rangle^\downarrow)^\uparrow && \text{(by } \langle \cdot \rangle \text{)} \end{aligned}$$

Note that the resulting interpretation has the form $\sim m^\uparrow$. Sequencing this result with the second conjunct, we continue calculating:

$$\begin{aligned} \langle \neg \exists x(\phi) \wedge \psi \rangle &= (\langle \neg \exists x(\phi) \rangle^\downarrow \wedge_\delta \langle \psi \rangle^\downarrow)^\uparrow && \text{(by } \langle \cdot \rangle \text{)} \\ &= ((\sim m^\uparrow)^\downarrow \wedge_\delta \langle \psi \rangle^\downarrow)^\uparrow && \text{(by above)} \\ &= (\neg_\delta m \wedge_\delta \langle \psi \rangle^\downarrow)^\uparrow && \text{(by Neg)} \end{aligned}$$

As this bears out, m ends up within the scope of δ 's static negation \neg_δ , and anaphora to $\exists x$ from within ψ is (correctly) predicted impossible.

3 Concrete instances

We began with a concrete dynamic semantics, DPL, which lacks DNE. In the previous section, we framed in maximally general or abstract terms the question of how to add DNE to a substrate dynamics that lacks it, i.e., without requiring anything beyond what's set out in Definitions 3 and 4. In this section, we'll fix δ to DPL, and consider various concrete Δ and $\langle \cdot \rangle$ that enrich δ with DNE (and nothing else, per Fact 2). In Section 4, we'll vary the substrate dynamics given by δ and $[\cdot]$.

Krahmer & Muskens (1995) give a theory of DNE by replacing DPL's interpretation function $[\cdot]$ with a pair of interpretation functions $[\cdot]^+$ and $[\cdot]^-$.⁵ The idea is that $[\phi]^+$ functions analogously to $[\phi]$, representing the 'positive' update or extension associated with ϕ , while $[\phi]^-$ functions analogously to $[\neg\phi]$, representing the 'negative' update or anti-extension associated with ϕ . Then negation simply toggles between positive and negative updates: $[\neg\phi]^+ := [\phi]^-$, and $[\neg\phi]^- := [\phi]^+$. DNE is immediate.

⁵ I take some liberties with my presentation of Krahmer & Muskens 1995. In their account, $[\cdot]$ interprets Discourse Representation Structures (DRS's), not first-order formulas, and DRS's are distinguished syntactically and semantically from static conditions.

Although Krahmer & Muskens define $[\cdot]^+$ and $[\cdot]^-$ directly, their semantics is easily recast as a lifted interpretation. Take $\delta ::= \text{DPL} ::= i \rightarrow i \rightarrow t$ with $[\cdot]$ as in Definition 1. Then set Δ to be *pairs* of DPL updates, $\Delta ::= \delta \times \delta$. Finally, define \uparrow , \downarrow and \sim as in Instance 1. \uparrow lifts a DPL meaning into Δ by pairing it with its DPL-negation; \downarrow takes us back into DPL by forgetting the anti-extension n ; and negation swaps the first and second coordinates of any pair in Δ .

Instance 1 (2D DPL). $\Delta ::= \delta \times \delta$ (pairs of updates).

$$m^\uparrow := (m, \neg_\delta m) \quad (m, n)^\downarrow := (m, n)^+ := m \quad \sim(m, n) := (n, m)$$

Instance 1 is plainly lawful. E.g., Neg: $(\sim m^\uparrow)^\downarrow = (\sim(m, \neg_\delta m))^\downarrow = (\neg_\delta m, m)^\downarrow = \neg_\delta m$. Moreover, the lifted interpretation $\langle \cdot \rangle$ yielded by Instance 1 is faithful to Krahmer & Muskens (modulo disjunction, which they treat statically and which we have not yet discussed; see Section 4). It is instructive to consider conjunction explicitly. The result is equivalent to the definition given by Krahmer & Muskens (1995: 366):

$$\begin{aligned} \langle \phi \wedge \psi \rangle &= (\langle \phi \rangle^\downarrow \wedge_\delta \langle \psi \rangle^\downarrow)^\uparrow && \text{(by } \langle \cdot \rangle \text{)} \\ &= (\langle \phi \rangle^+ \wedge_\delta \langle \psi \rangle^+)^\uparrow && \text{(def. of } \downarrow \text{)} \\ &= (\langle \phi \rangle^+ \wedge_\delta \langle \psi \rangle^+, \neg_\delta(\langle \phi \rangle^+ \wedge_\delta \langle \psi \rangle^+)) && \text{(def. of } \uparrow \text{)} \end{aligned}$$

A second, apparently quite different, theory of dynamic DNE is given by Gotham 2019 (see also Ranta 1994: 74f). Although in DPL $\neg\neg\phi \neq \phi$, Gotham points out that if $\neg\neg\phi$ is conjoined with a certain **dynamic tautology**, the result is equivalent to ϕ :

$$\neg\neg\phi \wedge (\phi \cup \neg\phi) \equiv \phi$$

The \cup here expresses the union of two updates, as in Definition 6. This is the externally dynamic ‘program disjunction’ of Groenendijk & Stokhof 1991a; Stone 1992:

Definition 6 (Program disjunction).

$$i[\phi \cup \psi] := i[\phi] \cup i[\psi]$$

For a dynamic tautology $\exists x(\phi) \cup \neg\exists x(\phi)$, the outputs at an input i will either be indices mapping x to $d \in D$ that witness the truth of ϕ (if $\exists x(\phi)$ is true), or the input i (if $\neg\exists x(\phi)$ is true).⁶ If we conjoin a doubly-negated existential claim with this dynamic tautology, as in $\neg\neg\exists x(\phi) \wedge (\exists x(\phi) \cup \neg\exists x(\phi))$, the left conjunct, though static, entails the truth of $\exists x(\phi)$. This commits us to left horn of the dynamic tautology. Thus, even though the left conjunct introduces no discourse referents on its own, it guarantees that the dynamic tautology in the right conjunct will.

⁶ One of these cases necessarily obtains, which is why formulas like $\phi \cup \neg\phi$ are dynamic tautologies.

How can this insight be operationalized? Gotham proposes that these dynamic tautologies are *projective*, i.e., that they live in a separate dimension from regular, at-issue meaning (cf. Potts 2005). At-issue content, including negation, is compositionally integrated in the at-issue dimension, leaving dynamic tautologies untouched. In effect, dynamic tautologies take wide scope over at-issue meaning. Finally, the two dimensions are dynamically conjoined. For doubly-negated sentences, this results in logical forms like $\neg\neg\phi \wedge (\phi \cup \neg\phi)$, which are equivalent to ϕ .

Like Krahmer & Muskens’ semantics, we can recast Gotham’s proposal as a lifted interpretation. The definitions of \uparrow , \downarrow , and \sim are given in Instance 2. As before, we take $\delta ::= \text{DPL}$, and $\Delta ::= \delta \times \delta$. But this time, \uparrow lifts DPL meanings m into Δ by pairing their double-negations, or **static closures**, with a dynamic tautology $m \cup \neg_\delta m$; \downarrow reconciles the two dimensions by conjoining them, and \sim maps DPL’s negation \neg_δ over the first, ‘at-issue’ coordinate of a two-dimensional meaning.⁷

Instance 2 (Decomposed updates). $\Delta ::= \delta \times \delta$ (pairs of updates, again).

$$m^\uparrow := (\neg_\delta \neg_\delta m, m \cup \neg_\delta m) \quad (m, n)^\downarrow := m \wedge_\delta n \quad \sim(m, n) := (\neg_\delta m, n)$$

Why does this work? \uparrow **decomposes** a DPL update into two pieces: the first, doubly-negated, half embodies its truth-conditional content, and the second half is a dynamic tautology that is itself silent on truth or falsity, but knows how to update in either event. \downarrow **reconstitutes** a decomposed update. And \sim targets the truth-conditional component, leaving the dynamic tautology untouched. Again, Emb and Neg are straightforwardly satisfied. There is, however, a slight hiccup for Inv, in that there are $m : \delta$ for which $\sim\sim(m, n) = (\neg_\delta \neg_\delta m, n) \neq (m, n)$ (for example, $m = [\exists x]$). Nevertheless, for any ϕ , $\sim\sim\langle\phi\rangle = \langle\phi\rangle$. The reason is that the at-issue dimension of $\langle\phi\rangle$ introduces no discourse referents by construction (given \uparrow), and for such static meanings, \neg_δ is involutive.

Should we be disturbed at the hiccup with Inv? Not really. For one, we can weaken Inv so that it only constrains $\text{Im}(\langle\cdot\rangle)$, the **image** (or range) of the lifted interpretation $\langle\cdot\rangle$ ($\text{Im}(f) := \{f(x) \mid x \in \text{Dom}(f)\}$). More fundamentally, though, because the ‘at-issue’ meanings of Instance 2 are inherently static, there’s actually no reason to represent them as dynamic in the first place! A simple proposition, or set of indices, will do.

This leads to a third lifted interpretation, Instance 3, which decomposes DPL updates into a static proposition and a dynamic tautology. $\text{True}_\delta(m)$ is the propositional content of m , i.e., the indices where m succeeds, $\{i : i \mid m(i) \neq \emptyset\}$ (Definition 2). $m|_p$ is

⁷ We make a couple departures from the letter of Gotham’s theory. First, Gotham’s at-issue dimension is not inherently static or doubly-negated. This plays no role in his account and disrupts Inv (and may lead to complications for non-idempotent ϕ , that is, ϕ that are not equivalent to $\phi \wedge \phi$, cf. Gotham 2019: 146). Second, Gotham assumes that anaphora to doubly-negated indefinites is inherently *unique*, which eventually results in a somewhat different treatment of dynamic tautologies (one which disrupts Emb). Gotham himself notes a counterexample to uniqueness (his (26)) that he credits to Matt Mandelkern.

the relation m restricted to inputs in p , that is, $\lambda i.m(i)$ if $i \in p$, \emptyset otherwise. Negation is complementation on the static proposition. Emb, Inv, and Neg are satisfied, no caveats.

Instance 3 (Staged updates). $\Delta ::= (\mathbf{i} \rightarrow \mathbf{t}) \times \delta$ (pairs of static meanings and updates).

$$m^\uparrow := (\text{True}_\delta(m), m \cup \neg_\delta m) \quad (p, m)^\downarrow := m|_p \quad \sim(p, m) := (\bar{p}, m)$$

Instance 3 decomposes DPL meanings into **staged updates**. The truth-conditional content and update potential that together comprise a DPL meaning are dissociated. This allows a classical set-theoretic negation to operate solely on the truth-conditional content. A single negation switches us over to indices in which the negative horn of the dynamic tautology is true; a second negation toggles back to the positive horn. The staged update is un-staged by \downarrow , which restricts the dynamic tautology according to the truth-conditional content, restoring an un-staged DPL meaning.

We consider one final lifted interpretation in this section. As a first step, notice that for each of Instances 1–3, $\text{Im}(\langle \cdot \rangle)$ ranges over only a small portion of the enriched type Δ . In Instance 1, the two updates associated with any $\langle \phi \rangle$ are necessarily contraries. But $\delta \times \delta$ includes many pairs of DPL updates (m, n) , including ones where m and n are totally unrelated! For Instances 2 and 3, for any $\langle \phi \rangle$, the static meaning in the first coordinate and the dynamic tautology in the second are likewise systematically related to each other, in that they’re generated from the same DPL meaning; additionally, in both cases, the second coordinate is always a dynamic tautology. But both Δ ’s contain all kinds of meanings, including very many which violate one or both of these constraints. In each of our three instances, then, the enriched type Δ is notably ‘loose’, in that Δ contains a lot of cruft beyond what is necessary to extend the substrate with DNE.

While this looseness is not problematic, it’s natural to wonder what a Δ that more closely tracked the requisite structure imposed by Definitions 3 and 4 would look like. We are interested, in other words, in characterizing a minimal, maximally constrained, or **canonical** enriched type Δ . This leads to our final lifted interpretation, Instance 4. Here, we simply tag vanilla DPL updates with a boolean that indicates if the update is to be left alone when executing \downarrow (if \top), or negated using \neg_δ (if F). Then \sim simply toggles this flag between \top and F . Emb, Inv, and Neg follow immediately. Fact 3 shows that the enriched type Δ and the lifted interpretations yielded by $\langle \cdot \rangle$ correspond closely.⁸

Instance 4 (Canonical Δ). $\Delta ::= \mathbf{t} \times \delta$ (pairs of booleans and updates).

$$m^\uparrow := (\top, m) \quad (b, m)^\downarrow := \begin{cases} m & \text{if } b \\ \neg_\delta m & \text{otherwise} \end{cases} \quad \sim(b, m) := (\bar{b}, m)$$

Fact 3 (Canonicity of Instance 4). $\mathbf{t} \times \text{Im}([\cdot]) = \text{Im}(\langle \cdot \rangle)$.

⁸ Fact 3 mentions $\mathbf{t} \times \text{Im}([\cdot])$ rather than $\mathbf{t} \times \delta$. δ includes many irrelevant $\mathbf{i} \rightarrow \mathbf{i} \rightarrow \mathbf{t}$ values, ones that do not correspond to the DPL interpretation of any ϕ — that is, δ is (orthogonally) loose with respect to $[\cdot]$.

Proof. \supseteq follows from induction on ϕ . For \subseteq , define $\tilde{\phi}$ as ϕ with all occurrences of $\neg\psi$ replaced with $\neg\tilde{\psi} \wedge \mathbf{1}$. $\mathbf{1}$ is any static DPL tautology, i.e., such that $i[\mathbf{1}] = \{i\}$. $\tilde{\phi}$ is then $\neg\neg$ -free. By Fact 2.ii, $\langle \widetilde{\neg\psi} \rangle = \langle \neg\tilde{\psi} \wedge \mathbf{1} \rangle = (\top, \langle \neg\tilde{\psi} \rangle^\downarrow) = (\top, [\neg\tilde{\psi}]) = [\neg\psi]^\uparrow$. A routine induction shows that $\langle \tilde{\phi} \rangle = [\phi]^\uparrow$ in general. So for any $(b, [\phi]) \in \mathfrak{t} \times \text{Im}([\cdot])$, if $b = \top$, $(b, [\phi]) = \langle \tilde{\phi} \rangle$, and if $b = \text{F}$, $(b, [\phi]) = \langle \neg\tilde{\phi} \rangle$. Thus $\mathfrak{t} \times \text{Im}([\cdot]) \subseteq \text{Im}(\langle \cdot \rangle)$. \square

This demonstrates that the minimal amount of extra structure imposed on DPL by Definitions 3 and 4 is quite modest — literally only an additional **bit** of information. Notably, the proof of Fact 3 applies more generally than DPL. Indeed, it establishes that for any substrate δ where some formula $\mathbf{1}$ is a right identity for \wedge (i.e., $[\phi \wedge \mathbf{1}] = [\phi]$), the canonical/minimal enriched type Δ is $\mathfrak{t} \times \delta$ (clearly, Instance 4 works for any δ).⁹

We’ve now seen four instances of lifted interpretations that extend DPL with DNE. Which should we prefer? While there’s something to be said about the minimality and simplicity of Instance 4, in an important sense, the question is ill-posed. All lifted interpretations respecting Definitions 3 and 4, from those based on extravagant Δ like Instances 1 and 2 (and, to a lesser extent, 3), to those based on minimal Δ that precisely embody the structure of lifted interpretations, like Instance 4, are **isomorphic**:

Fact 4 (Isomorphism of lawful lifts). Fix a substrate δ . Given $(\Delta_1, \uparrow_1, \downarrow_1, \sim_1, \langle \cdot \rangle_1)$ and $(\Delta_2, \uparrow_2, \downarrow_2, \sim_2, \langle \cdot \rangle_2)$ respecting Definitions 3 and 4, $\langle \cdot \rangle_1 \cong \langle \cdot \rangle_2$, i.e., there are bijections $f : \text{Im}(\langle \cdot \rangle_1) \rightarrow \text{Im}(\langle \cdot \rangle_2)$ and $g : \text{Im}(\langle \cdot \rangle_2) \rightarrow \text{Im}(\langle \cdot \rangle_1)$ such that for any $\phi \in \mathcal{L}$:

$$f(\langle \phi \rangle_1) = \langle \phi \rangle_2 \qquad g(\langle \phi \rangle_2) = \langle \phi \rangle_1$$

The proof is in Appendix A. Fact 4 entails that lifted interpretations over a δ , though potentially quite different in appearance and emphasis, do not differ meaningfully.

4 Varying the substrate

The architecture of lifted interpretations imposes minimal demands on the substrate dynamics. Per Definitions 3 and 4, the substrate need only define meanings for atomics and $\exists x$, and compositionally interpret \wedge and \neg . The rest is up to you.

In fact, though the concrete instances defined previously were billed as extensions of DPL, the definitions themselves are to a great extent neutral with regards to the underlying substrate. Instances 2 and 3 require that the substrate permits a way to union updates, and Instance 3 requires that meanings in the substrate have propositional contents they can be sensibly restricted by. If these requirements are met, the remainder of these instances (and the entirety of Instances 1 and 4), can be immediately repurposed

⁹ Although canonical Δ ’s are not unique, Fact 4 establishes that all lawful lifts over a given substrate δ are isomorphic. Given two canonical types $\Delta_1 = \text{Im}(\langle \cdot \rangle_1)$ and $\Delta_2 = \text{Im}(\langle \cdot \rangle_2)$ (fixing δ), since $\text{Im}(\langle \cdot \rangle_1) \cong \text{Im}(\langle \cdot \rangle_2)$ by Fact 4, it follows that $\Delta_1 \cong \Delta_2$. Hence, canonical Δ ’s are unique up to isomorphism.

for any dynamic substrate (assuming the substrate is consistent with Definitions 3 and 4). This includes update systems (Groenendijk & Stokhof 1991b), where dynamic meanings are typed as update functions on contexts (rather than relations on indices), systems that encode novelty for indefinites and familiarity for definites via partiality (Heim 1982; Dekker 1996), plural dynamic systems (van den Berg 1996; Nouwen 2003; Brasoveanu 2007), and many more.

Because lifted interpretations are conservative modulo DNE, whatever choices the substrate makes about lexical meaning automatically carry over to the lifted interpretation. The substrate may include weak and strong conditionals, generalized quantifiers and disjunctions (Groenendijk & Stokhof 1991a; Chierchia 1992; Kanazawa 1994). Some possible treatments of disjunction are as follows:¹⁰

$$\begin{aligned}\phi \vee \psi &:= \neg(\neg\phi \wedge \neg\psi) && \text{(strong, externally static)} \\ \phi \otimes \psi &:= \phi \vee (\neg\phi \wedge \psi) && \text{(weak, externally static)} \\ \phi \oplus \psi &:= \phi \cup (\neg\phi \wedge \psi) && \text{(weak, externally dynamic)}\end{aligned}$$

The point is not to decide between these treatments of disjunction, but to highlight how they interact with the basic outlook and formal results here. Lifted interpretations $\langle \phi \vee \psi \rangle$ and $\langle \phi \otimes \psi \rangle$ follow immediately from Definition 4. Only $\langle \phi \oplus \psi \rangle$, which relies on \cup (discussed previously, but not as part of the object language), requires us to say anything additional about $\langle \cdot \rangle$. The strategy for doing so is identical to $\langle \phi \wedge \psi \rangle$:

$$\langle \phi \cup \psi \rangle := (\langle \phi \rangle^\downarrow \cup_\delta \langle \psi \rangle^\downarrow)^\uparrow$$

As before, \cup_δ stands for whatever function is used to compositionally interpret \cup (e.g., for DPL, \cup_δ is \cup). Facts 1–4 still hold for the extended $\langle \cdot \rangle$, and their proofs are basically unchanged: the inductions for $\phi \cup \psi$ parallel those for $\phi \wedge \psi$, replacing $\wedge_{(\delta)}$ with $\cup_{(\delta)}$. More generally, the scheme for lifted interpretations is invariant for non-negated wff:

$$[\Sigma(\phi_1, \dots, \phi_k)] = \mu_\delta([\phi_1], \dots, [\phi_k]) \vdash \langle \Sigma(\phi_1, \dots, \phi_k) \rangle = (\mu_\delta(\langle \phi_1 \rangle^\downarrow, \dots, \langle \phi_k \rangle^\downarrow))^\uparrow$$

That is, if $[\phi]$ is compositional (again, supposing $\phi \neq \neg\psi$), the lifted interpretation $\langle \phi \rangle$ follows, together (mutatis mutandis) with proofs of Facts 1–4.¹¹

5 Bounded meaning

Mandelkern (2022) articulates a theory of anaphora that pairs static, propositional meaning with a second, projective dimension of meaning called a **bound**. A bounded

¹⁰ Here, ‘externally static’ characterizes (somewhat nonstandardly) ϕ which do not generate any discourse referents accessible to ψ in $\phi \wedge \psi$. Note that DNE allows certain ϕ and ψ in $\neg(\phi \vee \psi) \wedge \chi$ to bind into χ .

¹¹ Although quantifiers are often treated non-compositionally, in that meanings for quantified ϕ are not given in terms of an operation μ_δ operating over the meanings of ϕ ’s parts, compositional treatments of quantification are well known, and natural in dynamic settings (e.g., Kobele 2010).

meaning for a simple existential formula $\exists x(Fx)$ is comprised of the usual static content, paired with a bound requiring that if anything is F , x is:

$$\underbrace{\exists x(Fx)}_{\text{regular meaning}} \qquad \underbrace{\exists x(Fx) \rightarrow Fx}_{\text{bound}}$$

On the bounded theory, indefinites have two faces. In regular meanings, indefinites are static existential quantifiers. In bounds, indefinites are quasi-variables.¹² Crucially, bounds are **projective**. Existentially quantified formulas $\exists x(\phi)$ introduce a bound $\exists x(\phi) \rightarrow \phi$. This bound is (simplifying) inherited by any formulas that embed $\exists x(\phi)$.

Although neither side of a bounded meaning treats $\exists x$ dynamically, characteristically dynamic patterns of anaphora are nevertheless predicted when regular meanings and bounds are brought together. In (2), for example, conjoining the regular meaning $\exists x(Lx \wedge Wx) \wedge Sx$ with the bound $\exists x(Lx \wedge Wx) \rightarrow (Lx \wedge Wx)$ (introduced by the first conjunct and projected up) yields $Lx \wedge Wx \wedge Sx$. If x is novel (that is, nothing is presupposed about its value, see fn 12), this amounts to a claim that something is L , W , and S .

DNE follows from the bounded theory because negation operates classically on regular meanings, and bounds project past negation. Partee disjunctions like (7) work, as well: conjoining the regular meaning $\neg \exists x(Cx) \vee Bx$ and the bound $\exists x(Cx) \rightarrow Cx$ (triggered by $\exists x(Cx)$, then projected up past \neg and \vee) yields $\neg \exists x(Cx) \vee (Cx \wedge Bx)$. This meaning is weak: assuming x is novel, it says that there are no C 's, or that a C is B .

Indeed, the bounded theory exclusively derives weak meanings when variables are valued via bounds. Spector (2025) highlights *it isn't true that Sue has donkey and that she beats it*, with regular meaning $\neg(\exists x(Dx) \wedge Bx)$ and bound $\exists x(Dx) \rightarrow Dx$. Their conjunction is equivalent to $\neg \exists x(Dx) \vee (Dx \wedge \neg Bx)$. If x is novel, this amounts, counterintuitively, to a claim that Sue has no donkeys, or doesn't beat all her donkeys.

The bounded theory is implemented with an interpretation function that associates formulas with two booleans relative to a context (a set of assignment-world pairs), an assignment, and a world. Regular meanings are sensitive to g and w ; bounds may be sensitive to c , as well. A bounded extension for $\exists x(Fx)$ is below. A variable x is Novel in c iff c includes every possible value for x , i.e., if $\{g(x) \mid (g, w) \in c\} = \mathcal{D}_e \cup \{\#\}$.¹³

$$\langle \exists x(Fx) \rangle^{c,g,w} = (\llbracket \exists x(Fx) \rrbracket^{g,w}, \text{Novel}(x, c) \wedge \llbracket \exists x(Fx) \rightarrow Fx \rrbracket^{g,w})$$

The **update** of c with ϕ is then given by $c\langle\phi\rangle := \{(g, w) \in c \mid \langle\phi\rangle^{c,g,w} = (\top, \top)\}$. This brings regular meanings and bounds together, exactly as if they were conjoined.

¹² The treatment of indefinites as (quasi)-variables dates back to Heim (1982: Ch. 2) (see also Dekker 1996; Rothschild 2017). On such approaches, discourse referents are introduced not by modifying input assignments (as they are in DPL), but by eliminating assignments that associate the variable in question with undefinedness/#. It is critical in such approaches to ensure novelty, that is, to never coindex two indefinites: on variabilist theories, coindexed indefinites may end up problematically *covalued*.

¹³ Mandelkern uses c to encode familiarity of definites, rather than novelty of indefinites. My aim here is only to illustrate how the context parameter c can be put to use, and I set familiarity aside.

So, assuming novelty, $c\langle\exists x(Fx)\rangle = \{(g, w) \in c \mid \llbracket Fx \rrbracket^{g,w}\}$, $c\langle\exists x(Lx \wedge Wx) \wedge Sx\rangle = \{(g, w) \in c \mid \llbracket Lx \wedge Wx \wedge Sx \rrbracket^{g,w}\}$, etc. I don't include clauses for negation, conjunction, or disjunction. It suffices to note that their regular meanings are classical, and that they project bounds in a way consistent with the foregoing discussion.

How do bounded meanings relate to lifted interpretations? We begin by considering the *intensions* delivered by the bounded interpretation function. Because $\langle\phi\rangle^{c,g,w} : \mathfrak{t} \times \mathfrak{t}$, $\langle\phi\rangle$ simpliciter has type $c \times \mathfrak{i} \rightarrow \mathfrak{t} \times \mathfrak{t}$, where \mathfrak{i} is, as before, the type of indices (here, world-assignment pairs), and c is the type of a context or set of indices, $\mathfrak{i} \rightarrow \mathfrak{t}$. This type is isomorphic to $(c \times \mathfrak{i} \rightarrow \mathfrak{t}) \times (c \times \mathfrak{i} \rightarrow \mathfrak{t})$, which in turn is isomorphic to $(c \rightarrow \mathfrak{i} \rightarrow \mathfrak{t}) \times (c \rightarrow \mathfrak{i} \rightarrow \mathfrak{t})$, which is just another name for $(c \rightarrow c) \times (c \rightarrow c)$. Now, as mentioned previously, only the bound is sensitive to the context parameter. This means that the left $c \rightarrow c$ coordinate actually only includes *constant* functions, and can therefore be identified with c . Altogether, bounded intensions are type $c \times (c \rightarrow c)$.

This looks remarkably like the type of staged updates (Instance 3), although the bound is typed as an update function on contexts (type $c \rightarrow c$) rather than a relation on indices (type $\text{DPL} = \mathfrak{i} \rightarrow c$).¹⁴ The higher type owes to the fact that the bounded theory imposes global constraints on input contexts (novelty/familiarity), which cannot be reduced to distributive updates on individual indices (Groenendijk & Stokhof 1991b). As pointed out in Section 4, Instance 3 could easily be defined over an update-theoretic substrate, in which case the types would align perfectly.

The similarity of the bounded theory and theories with staged updates is even more striking when we consider that the bounds associated with existential quantifiers, $\exists x(\phi) \rightarrow \phi$, are classically equivalent to $\phi \vee \neg\exists x(\phi)$. Modulo the quasi-variabilist take on indefinites, this bound is akin to the dynamic tautologies from Instances 2 and 3. Moreover, the regular half of a bounded meaning plays an analogous role to the static propositional meanings that ground staged update systems.

Is the bounded theory just a system of staged updates? On the one hand, both the bounded theory and Instance 3 replace a single dynamic meaning with a pair of a static meaning and a (nearly) tautological update, with negation operating classically on the static component. On the other, it is not clear how to formulate the bounded theory *as a lifted interpretation* consistent with Definitions 3 and 4 (other than trivially, that is, with $\uparrow = \downarrow = \text{id}$). One stumbling block is that existential quantification is treated statically in the regular coordinate of a bounded meaning — e.g., in $\exists x(Lx \wedge Wx) \wedge Sx$, the latter x is free. This choice is responsible for the bounded theory's persistent gener-

¹⁴ I should point out that nothing in the bounded theory's definition of $\langle\phi\rangle^{c,g,w}$ guarantees that $(g, w) \in c$, which seems necessary for the $c \rightarrow c$ bound to be seen as a legible update function on contexts. Still, it's reasonable to restrict our attention to such "proper" points of evaluation — adopting the terminology of Kolodny & MacFarlane (2010), who make a related point about Yalcin's (2007) recasting of Veltman's (1996) update semantics. Proper points of evaluation correspond to the information states of actual communicative agents; the update rule $c\langle\phi\rangle := \{(g, w) \in c \mid \langle\phi\rangle^{c,g,w} = (T, T)\}$ codifies this restriction.

ation of weak meanings (as unbound variables are scopeless), and it is at odds with the basic idea of lifted interpretations — that they *extend* an underlying *dynamic* substrate.

A second difference is seen in the way dynamic tautologies project. The bound of $\exists x(Fx) \wedge Gx$, $\exists x(Fx) \rightarrow Fx$, is identical to the bound of $\exists x(Fx)$. In a staged update, by contrast, the dynamic tautology for $\exists x(Fx) \wedge Gx$ is based on the entire conjunction, not on $\exists x(Fx)$ alone. (By Fact 2.i, $\langle \exists x(Fx) \wedge Gx \rangle = [\exists x(Fx) \wedge Gx]^\uparrow$.) The reason for this difference is that lifted interpretations rely on the substrate's \wedge_δ , and \downarrow must apply to the lifted interpretations of the individual conjuncts before they can be sequenced with \wedge_δ . Staged updates are constantly unstaged in the course of lifted interpretation.

Staged updates and bounded meanings have deep genetic (and empirical) similarities, while representing importantly different choices about whether to accept dynamic binding, as well as the degree to which dynamic tautologies are projective. Eschewing dynamic binding is motivated to the extent that pervasively weak meanings are deemed desirable. Treating dynamic tautologies as projective has less empirical bite, but is motivated to the extent that one wishes to validate equalities like $\phi \equiv \phi \wedge \mathbf{1}$ (as noted in the proof of Fact 3, these formulas are not always equivalent in lifted interpretations).

Lifted interpretations are motivated to the extent that one wishes to preserve the structure and insights of an underlying dynamic semantics, while benefiting from dynamic negation. Whereas the bounded theory rolls its own semantics for quantifiers, one which relies on domain variables and careful consideration of quantificational bounds, lifted interpretations simply and modularly inherit the substrate's treatment of quantification. Finally, lifted interpretations can serve as a useful starting point in theory-building. Once an instance is characterized, other treatments of $\langle \phi \wedge \psi \rangle$ may be entertained. Alternatively, lifted interpretations may interact in non-trivial ways with other aspects of a compositional semantics, resulting in systems that organically extend lifted interpretations beyond dynamic negation. We turn to this latter possibility now.

6 Exceptional scope

In earlier work (Charlow 2014, 2025), I have argued for a version of dynamic semantics that simultaneously explains the ability of indefinites to dynamically bind, and to take exceptional *quantificational* scope out of islands (Fodor & Sag 1982; Reinhart 1997):

(10) If {a, every} rich relative of mine dies, I'll inherit a house. $\{\exists, *\forall\} \gg$ if

The theory is based on a generalization of dynamic semantics that allows any value of any type to incur dynamic effects. The key pieces are summarized in Definition 7. A value of type D_σ reads in an input index and nondeterministically returns a σ , together with a possibly modified output index. $\eta : \sigma \rightarrow D_\sigma$ upgrades non-dynamic meanings into trivially dynamic ones. $\star : D_\sigma \rightarrow (\sigma \rightarrow D_\tau) \rightarrow D_\tau$ composes dynamic meanings.

Definition 7 (Dynamic semantics with exceptional scope).

$$D_\sigma ::= i \rightarrow (\sigma \times i) \rightarrow t \quad \eta(x) := \lambda i. \{(x, i)\} \quad m \star f := \lambda i. \bigcup_{(x, j) \in m(i)} f(x)(j)$$

The mapping \star is **associative** in the sense of Fact 5, which means that m may appear to scope (exceptionally) over g , even when m actually only scopes (locally) over f :

Fact 5 (Associativity of \star). $(m \star \lambda x. f(x)) \star g = m \star (\lambda x. f(x) \star g)$

Staged updates are type $(i \rightarrow t) \times \text{DPL}$, which is isomorphic to $i \rightarrow t \times (i \rightarrow t)$. This is highly reminiscent of $D_t = i \rightarrow (t \times i) \rightarrow t$, but D_t is *richer*. Specifically, D_t includes M such that $M(i) = \{(T, i^{x \rightarrow a}), (F, i^{x \rightarrow b})\}$, i.e., with **value nondeterminism**. In other words, for such M , the t in D_t ranges over more than one value (T and F). Such M arise naturally in the semantics and are central to its account of exceptional scope as nondeterminism projected via \star , but are impossible to represent in $i \rightarrow t \times (i \rightarrow t)$ (since the boolean value, the leftmost t , is completely determined given the input i).

Because D_t is richer than staged updates, it is no surprise that we can characterize a lifted interpretation based on it, as in Instance 5. This interpretation doesn't exploit the richness of D_t (nor could it, given Fact 4). For any ϕ , i , and $(b, j) \in i\langle\phi\rangle$, either $b = T$ or $b = F$, without value nondeterminism, and exactly like $i \rightarrow t \times (i \rightarrow t)$.

Instance 5 (Lifted interpretation for D_t). $\Delta ::= D_t = i \rightarrow (t \times i) \rightarrow t$.

$$m^\uparrow := \lambda i. \begin{cases} \{(T, j) \mid j \in m(i)\} & \text{if } \text{True}_\delta(m, i) \\ \{(F, i)\} & \text{otherwise} \end{cases} \quad M^\downarrow := \lambda i. \{j \mid (T, j) \in M(i)\}$$

$$\sim M := \lambda i. \{(\bar{b}, j) \mid (b, j) \in M(i)\}$$

Having characterized this instance, we set it aside. Instead, we will consider how to view D_t as an *enrichment of staged updates*, one that adds the ability to represent value nondeterminism, and thus to explain exceptional scope. Given $\Delta ::= i \rightarrow t \times (i \rightarrow t)$, we define $\uparrow : \Delta \rightarrow D_t$ and $\downarrow : D_t \rightarrow \Delta$ as below. Note that $\langle\phi\rangle^{\uparrow\downarrow} = \langle\phi\rangle$ generally:

Definition 8 (Mapping between staged updates and D_t).

$$M^\uparrow := \lambda i. \{(b, j) \mid (b, J) = M(i), j \in J\}$$

$$\mathbf{M}^\downarrow := \lambda i. \begin{cases} (\text{True}_D(\mathbf{M}, i), \{j \mid (\text{True}_D(\mathbf{M}, i), j) \in \mathbf{M}(i)\}) & \text{if } \mathbf{M}(i) \neq \emptyset \\ (F, \{i\}) & \text{otherwise} \end{cases}$$

\uparrow lifts M into D_t by distributing its boolean (at an input) to each of its outputs. \downarrow lowers \mathbf{M} onto Δ by producing a summary boolean value $\text{True}_D(\mathbf{M}, i) := \exists j : (T, j) \in \mathbf{M}(i)$, and only keeping j from $(b, j) \in \mathbf{M}(i)$ if b agrees with this value (the second horn of

the definition guarantees that when \mathbf{M} fails, \mathbf{M}^\downarrow still has an output; the outputs of a staged update owe to a dynamic tautology, so failure isn't an option).¹⁵

Definition 9 gives a semantics for indefinites that may yield value nondeterminism. The alternatives witnessing the truth of the restrictor ϕ are used to evaluate the scope ψ ($\text{guard}(b) := \eta(b)$ if b , $\lambda i.\emptyset$ otherwise). These alternatives may make ψ wholly true, wholly false, or a nondeterministic mix. If a is a linguist who walked in the park at i and b is a linguist who didn't, $i\langle \mathcal{E}x(Lx, Wx) \rangle = \{(\top, i^{x \mapsto a}), (\text{F}, i^{x \mapsto b})\}$. This meaning can't be faithfully represented in Δ : $i\langle \mathcal{E}x(Lx, Wx) \rangle^\downarrow = (\top, \{i^{x \mapsto a}\})$.

Definition 9 (Exceptional indefinites).

$$\begin{aligned} \langle \mathcal{E}x(\phi, \psi) \rangle &:= \langle \exists x(\phi) \rangle^\uparrow \star \lambda b.\text{guard}(b) \star \lambda _.\langle \psi \rangle^\uparrow \\ &= \lambda i. \bigcup_{(\top, j) \in i\langle \exists x(\phi) \rangle^\uparrow} j\langle \psi \rangle^\uparrow \end{aligned}$$

Because this definition has the shape $\mathbf{M} \star f$, it is subject to the associativity of \star , and therefore consistent with the \star -based theory of exceptional scope. This contrasts with prior attempts to exploit D_t to give an account of DNE (Elliott 2020), which jettison the D_t account of exceptional scope. Up against space constraints and, regrettably, without a typed compositional semantics to lean on, that is where I'll have to leave it, for now.

7 Conclusion

Lifted interpretations are a useful framework for approaching dynamic negation. They help us better understand existing theories, revealing similarities (and in some cases isomorphisms) between accounts of DNE that appear superficially quite different. Lifted interpretations lead naturally to revealing and useful abstractions like staged update systems, which have deep connections with recent work on bounded meaning. Overall, then, lifted interpretations suggest that disparate theories of DNE in dynamic semantics work for surprisingly similar reasons.

In future work, I hope to extend this program to other accounts not discussed in this paper. This includes recent work by Hofmann (2019, 2025), which appears to rely on meanings richer than staged updates, and Spector (2025), which relies on more constrained (indeed, static) meanings in a system similar to bounded meanings.

Taking this perspective also suggests new territory to explore. D-based theories of exceptional indefinite scope are properly richer than staged updates. In the last section, we sketched how to view D_t meanings as an enrichment of staged updates, allowing dynamic binding, DNE, and exceptional scope to be explained within a single system, and with a precise sense of which enrichments are responsible for which capabilities.

¹⁵ We have not actually defined a lifted interpretation for $\Delta ::= i \rightarrow t \times (i \rightarrow t)$. The requisite instance is isomorphic to Instance 3, with $\lambda i.(\alpha, \beta)$ replacing $(\lambda i.\alpha, \lambda i.\beta)$, and $M^\downarrow := \lambda i.\{j \in J \mid (T, J) = M(i)\}$.

A Proofs

Proof of Fact 2 (Lifted interpretations are conservative over $[\cdot]$).

Part (i). For \neg -free ϕ , $\langle \phi \rangle = [\phi]^\uparrow$. By induction on \neg -free ϕ :

- ▷ *Base case* (atomic or $\exists x$): $\langle \phi \rangle = [\phi]^\uparrow$ by definition of $\langle \cdot \rangle$.
- ▷ *Inductive case* ($\phi = \psi \wedge \chi$):

$$\begin{aligned}
 \langle \psi \wedge \chi \rangle &= (\langle \psi \rangle^\downarrow \wedge_\delta \langle \chi \rangle^\downarrow)^\uparrow && \text{(by } \langle \cdot \rangle \text{)} \\
 &= ([\psi]^\uparrow \wedge_\delta [\chi]^\uparrow)^\uparrow && \text{(by IH)} \\
 &= ([\psi] \wedge_\delta [\chi])^\uparrow && \text{(by Emb)} \\
 &= [\psi \wedge \chi]^\uparrow && \text{(by } [\cdot] \text{)}
 \end{aligned}$$

Part (ii): For $\neg\neg$ -free ϕ , $\langle \phi \rangle^\downarrow = [\phi]$. By induction on $\neg\neg$ -free ϕ :

- ▷ *Base case* (ϕ atomic or $\exists x$): $\langle \phi \rangle^\downarrow = [\phi]^\uparrow \downarrow = [\phi]$ by $\langle \cdot \rangle$ and Emb.
- ▷ *Case* ($\phi = \psi \wedge \chi$):

$$\begin{aligned}
 \langle \psi \wedge \chi \rangle^\downarrow &= (\langle \psi \rangle^\downarrow \wedge_\delta \langle \chi \rangle^\downarrow)^\uparrow \downarrow && \text{(by } \langle \cdot \rangle \text{)} \\
 &= \langle \psi \rangle^\downarrow \wedge_\delta \langle \chi \rangle^\downarrow && \text{(by Emb)} \\
 &= [\psi] \wedge_\delta [\chi] && \text{(by IH)} \\
 &= [\psi \wedge \chi] && \text{(by } [\cdot] \text{)}
 \end{aligned}$$

- ▷ *Case* ($\phi = \neg\psi$): Since ϕ is $\neg\neg$ -free, ψ is not of the form $\neg\chi$.

- ▷ If ψ is atomic or $\exists x$: $\langle \neg\psi \rangle^\downarrow = (\sim \langle \psi \rangle)^\downarrow = (\sim [\psi]^\uparrow)^\downarrow = \neg_\delta [\psi] = [\neg\psi]$.
- ▷ If $\psi = \chi \wedge \xi$ (χ and ξ are $\neg\neg$ -free):

$$\begin{aligned}
 \langle \neg(\chi \wedge \xi) \rangle^\downarrow &= (\sim \langle \chi \wedge \xi \rangle)^\downarrow && \text{(by } \langle \cdot \rangle \text{)} \\
 &= (\sim (\langle \chi \rangle^\downarrow \wedge_\delta \langle \xi \rangle^\downarrow)^\uparrow)^\downarrow && \text{(by } \langle \cdot \rangle \text{)} \\
 &= (\sim ([\chi] \wedge_\delta [\xi])^\uparrow)^\downarrow && \text{(by IH)} \\
 &= \neg_\delta ([\chi] \wedge_\delta [\xi]) && \text{(by Neg)} \\
 &= [\neg(\chi \wedge \xi)] && \text{(by } [\cdot] \text{)} \quad \square
 \end{aligned}$$

Remark. Fact 2.ii entails that for any ϕ which is $\neg\neg$ -free and which is not itself of the form $\neg\psi$, $\langle \phi \rangle = [\phi]^\uparrow$. The base case is immediate. For $\phi = \psi \wedge \chi$:

$$\langle \psi \wedge \chi \rangle = (\langle \psi \rangle^\downarrow \wedge_\delta \langle \chi \rangle^\downarrow)^\uparrow = ([\psi] \wedge_\delta [\chi])^\uparrow = [\psi \wedge \chi]^\uparrow \quad \text{by Fact 2.ii}$$

So Fact 2.i follows from Fact 2.ii. It also follows that for such ϕ , $\langle \phi \rangle^{\downarrow\uparrow} = \langle \phi \rangle = [\phi]^\uparrow$.

Proof of Fact 4 (Isomorphism of lifts). Fix a substrate δ . Given $(\Delta_1, \uparrow_1, \downarrow_1, \sim_1, \langle \cdot \rangle_1)$ and $(\Delta_2, \uparrow_2, \downarrow_2, \sim_2, \langle \cdot \rangle_2)$ respecting Definitions 3 and 4, $\langle \cdot \rangle_1 \cong \langle \cdot \rangle_2$ via f :

$$f : \text{Im}(\langle \cdot \rangle_1) \rightarrow \text{Im}(\langle \cdot \rangle_2) \quad f(m^{\uparrow_1}) = m^{\uparrow_2} \quad f(\sim_1 M) = \sim_2 f(M)$$

Per Definition 4, f exhaustively covers all $M \in \text{Im}(\langle \cdot \rangle_1)$. Therefore f is total. By Emb, \uparrow_1 is injective. By Inv, \sim_1 is injective. So f is well-defined wherever the outputs of \uparrow_1 and \sim_1 do not coincide. If $M \in \text{Im}(\langle \cdot \rangle_1)$ satisfies both $M = m^{\uparrow_1}$ and $M = \sim_1 N$ for some m and N , then by Inv, $N = \sim_1 m^{\uparrow_1}$. In that case:

$$f(\sim_1 N) = \sim_2 f(N) = \sim_2 f(\sim_1 m^{\uparrow_1}) = \sim_2 \sim_2 f(m^{\uparrow_1}) = f(m^{\uparrow_1})$$

The defining clauses agree on $f(M)$. So f is well-defined.

Claim. For all $M \in \text{Im}(\langle \cdot \rangle_1)$, $f(M)^{\downarrow_2} = M^{\downarrow_1}$.

Proof of claim. Every $M \in \text{Im}(\langle \cdot \rangle_1)$ is generated from an m^{\uparrow_1} by iterated applications of \sim_1 ; hence (given Inv) M is equivalent to m^{\uparrow_1} or $\sim_1 m^{\uparrow_1}$. We reason by cases:

- ▷ *Case* ($M = m^{\uparrow_1}$): $f(M)^{\downarrow_2} = m^{\uparrow_2 \downarrow_2} = m = M^{\downarrow_1}$ by Emb.
- ▷ *Case* ($M = \sim_1 m^{\uparrow_1}$): $f(M)^{\downarrow_2} = (\sim_2 m^{\uparrow_2})^{\downarrow_2} = \neg_\delta m = (\sim_1 m^{\uparrow_1})^{\downarrow_1} = M^{\downarrow_1}$ by Neg.

Main result. We show $f(\langle \phi \rangle_1) = \langle \phi \rangle_2$ by induction on ϕ .

- ▷ *Base case* (ϕ atomic or $\exists x$): $f(\langle \phi \rangle_1) = f([\phi]^{\uparrow_1}) = [\phi]^{\uparrow_2} = \langle \phi \rangle_2$.
- ▷ *Case* ($\phi = \psi \wedge \chi$):

$$\begin{aligned} f(\langle \psi \wedge \chi \rangle_1) &= f((\langle \psi \rangle_1^{\downarrow_1} \wedge_\delta \langle \chi \rangle_1^{\downarrow_1})^{\uparrow_1}) && \text{(by } \langle \cdot \rangle) \\ &= (\langle \psi \rangle_1^{\downarrow_1} \wedge_\delta \langle \chi \rangle_1^{\downarrow_1})^{\uparrow_2} && \text{(def. of } f \text{ on } m^{\uparrow_1}) \\ &= (f(\langle \psi \rangle_1)^{\downarrow_2} \wedge_\delta f(\langle \chi \rangle_1)^{\downarrow_2})^{\uparrow_2} && \text{(by Claim)} \\ &= (\langle \psi \rangle_2^{\downarrow_2} \wedge_\delta \langle \chi \rangle_2^{\downarrow_2})^{\uparrow_2} && \text{(by IH)} \\ &= \langle \psi \wedge \chi \rangle_2 && \text{(by } \langle \cdot \rangle) \end{aligned}$$

- ▷ *Case* ($\phi = \neg \psi$):

$$f(\langle \neg \psi \rangle_1) = f(\sim_1 \langle \psi \rangle_1) = \sim_2 f(\langle \psi \rangle_1) = \sim_2 \langle \psi \rangle_2 = \langle \neg \psi \rangle_2 \quad \text{by IH.}$$

Define $g : \text{Im}(\langle \cdot \rangle_2) \rightarrow \text{Im}(\langle \cdot \rangle_1)$ symmetrically. Then $g(\langle \phi \rangle_2) = \langle \phi \rangle_1$, and so $g \circ f = \text{id}$, and $f \circ g = \text{id}$. □

Corollary. For all $\phi \in \mathcal{L}$, $\langle \phi \rangle_1^{\downarrow_1} = \langle \phi \rangle_2^{\downarrow_2}$. By the main result, $f(\langle \phi \rangle_1) = \langle \phi \rangle_2$. By the Claim, $f(\langle \phi \rangle_1)^{\downarrow_2} = \langle \phi \rangle_1^{\downarrow_1}$. Hence $\langle \phi \rangle_2^{\downarrow_2} = \langle \phi \rangle_1^{\downarrow_1}$.

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